Discrete Optimization and Network Analysis

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I Can Tell You My Secret Now?

I see graphs everywhere!
Graphs (Networks)

Graph $G=(V, E)$
- Vertex set $V$ is finite
- Edges $E = \{uv : u,v \in V\}$
- Undirected (for this talk)
- $u$ is a neighbor of $w$ if $uw \in E$
Networks are Everywhere!

Yang 2010

Carlson et al. 2006

Don Johnson, Rice
Who Likes Bacon?
Social Network Pop Quiz
9-11 Terrorist Network

1) Alshehri
2) Sugami
3) Al-Marabh
4) Hijazi
5) W. Alshehri
6) A. Alghamdi
7) M. Alshehri
8) S. Alghamdi
9) Ahmed
10) Al-Hisawi
11) Al-Omari
12) H. Alghamdi
13) Alnami
14) Al-Haznawi
15) Darkazanli
16) Abdi
17) Al-Shehhi
18) Essabar
19) S. Alhazmi
20) N. Alhazmi
21) Bahaji
22) Jarrah
23) Atta
24) Shaikh
25) El Motassadeq
26) Al-Mihdhar
27) Moussaoui
28) Al-Shibh
29) Raissi
30) Hanjour
31) Awadallah
32) Budiman
33) Al-ani
34) Moqed
35) Abdullah
36) Al Salmi
37) Alhazmi
Homer Ignoring Lisa
The Simpsons Social Network
What is cohesiveness in terms of graphs?

• Debated by social scientists
• Three general properties
  – Familiarity (few strangers)
  – Members can easily reach each other (quick communication)
  – Robustness (not easily destroyed by removing members)
Is this graph cohesive?

Clique is too restrictive!
Different versions of cohesiveness

- Relax distance requirement between members
  - $k$-clique (Luce 1950)
  - $k$-club (Alba 1973)
- Relax the familiarity (# of neighbors) between members
  - $k$-plex (Siedman & Foster 1978)
  - $k$-core (Siedman 1983)

Others: F-group, LS sets, Lambda sets, quasi-cliques, k-trusses
Talks at this Conference on Clique Generalizations

- “An Upper-bounding Technique for the Maximum Quasi-clique Problem” by Balasundaram and Miao
- “Algorithms for the Maximum Edge Weight Clique Problem” by Fontes, Butenko, and Hosseinian
- “Parsimonious Formulations for Low-Diameter Clusters” by Salemi and Buchanan
Clarity

CLARITY IS THE PATH TO INNER PEACE.
Outline

I. Definitions

II. My Research in the Area

III. Conclusions
Given a graph \( G=(V, E) \) and some integer \( k > 0 \), a set \( S \subseteq V \) is called a \( k \)-plex if every node of \( S \) has at most \( k-1 \) non-neighbors in \( S \).

- Cliques are 1-plexes.

- NP-hard to find maximum \( k \)-plex, \( \omega_k(G) \), in a graph \( G \).
1-plexes

1-plexes are cliques
2-plexes

at most 1 non-neighbor
9-11 Terrorist Network

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34) Moqed
35) Abdullah
36) Al Salmi
37) Alhazmi
Another Example: Retail Location

Stable set
Starbucks in Springfield
Another Example: Retail Location
Co-\(k\)-plexes

- A set \(S \subseteq V\) is called a **co-\(k\)-plex** if every node of \(S\) has at most \(k-1\) neighbors in \(S\).

- Stable sets are co-1-plexes.

- NP-hard to find maximum co-\(k\)-plex, \(\alpha_k(G)\) in a graph \(G\).

- Co-2-plexes correspond to vertex induced subgraphs of isolated nodes and matched pairs.
\(k\)-plexes and co-\(k\)-plexes are equivalent problems
Not k-clubbing!
Given a graph $G = (V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a $k$-club if $\text{diam}(G[S]) \leq k$.
K-cores

- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a $k$-core if every node of $G[S]$ has degree at least $k$.

3-core

NP-hard for minimum k-core
My Research: Analogs

Analogs in Combinatorial Algorithms
Analogs in Polyhedral Combinatorics
Analogs in Combinatorial Algorithms

- K-plexes (McClosky & Hicks, 2012)
- Co-plex coloring (McClosky & Hicks 2012; McClosky, Arellano & Hicks, 2015)
- K-cores (Wood & Hicks, 2015)
- Co-2-plexes in \{claw,bull\}-free graphs (Wood, McClosky & Hicks)
Graph Coloring

Clique number \( \omega(G) \leq \chi(G) \) Chromatic number
Co-\(k\)-plex Coloring

K-plex number \(\omega_k(G) \leq \chi_k(G)\)  Co-k-plex chromatic number

McClosky & Hicks (2012)
Co-k-plex Coloring Properties

• Thm (McClosky, Arellano & Hicks, 2015): For any graph $G$, $\chi_k(G) \leq k\chi(G)$.

• $M = (S, I)$ with finite set $S$ and family $I$ of subsets of $S$ is called a matroid if the following axioms are satisfied
  
  – $\emptyset \in I$
  
  – if $J' \subseteq J \in I$, then $J' \in I$
  
  – for every $A \subseteq S$, every maximal independent subset of $A$ has the same cardinality
Co-k-plex Coloring Properties

• A graph is **k-plex matroidal** if the \((V(G), K)\) is a matroid where \(K\) is the family of vertex subsets of \(G\) that form \(k\)-plexes.

• Thm (McClosky, Arellano & Hicks, 2015): If graph \(G\) is \(k\)-plex matroidal then \(\omega_k(G') = \chi_k(G')\) for all vertex induced subgraphs \(G'\)

• Also gave analog of Kubale & Jackowski (1985)
Claw-free graphs

- Minty (1980)
  - Combinatorial algorithm for claw free graphs
Two approaches

- Transformation from \{\text{claw, bull}\}-free to claw-free
- Direct algorithm
- Both polynomial-time combinatorial algorithms
Analogs in Polyhedral Combinatorics

- K-plex Polytope
- Co-2-plex Polytope
- 2-Club Polytope
Polyhedral Combinatorics Analogs for general $k$

• K-plex polytope (Balasundaram, Butenko & Hicks, 2011)
  - Maximal stable sets
  - Holes (certain types)
Analogons in Polyhedral Combinatorics for co-2-plex polytope

- Padberg (1973)
  - Clique and odd hole inequalities
- Trotter (1975)
  - Web inequalities
- Minty (1980)
  - Claw free graphs
2-plex Inequalities

- Thm (Padberg, 1973): If $K$ is a maximal clique in $G$, then $\sum_{v \in K} x_v \leq 1$ is a facet for stable set polytope of $G$.

- Thm [(McClosky & Hicks, 2009), (Balasundaram et al., 2011)]: If $K$ is a maximal 2-plex in $G$ such that $|K| > 2$, then $\sum_{v \in K} x_v \leq 2$ is a facet for co-2-plex polytope of $G$. 
Thm (Padberg, 1973): If $C$ is an $n$-chordless cycle such that $n > 3$ is odd, then $\sum_{v \in V(C)} x_v \leq \lfloor n/2 \rfloor$ is a facet for the stable set polytope.
Thm (McClosky & Hicks, 2009): If $C$ is an $n$-chordless cycle such that $n > 2$ and $n \not\equiv 0 \pmod{3}$, then $\sum_{v \in V(C)} x_v \leq \left\lfloor \frac{2n}{3} \right\rfloor$ is a facet for the co-2-plex polytope.
• For fixed integers \( n \geq 1 \) and \( p \) such that \( 1 \leq p \leq \lfloor n/2 \rfloor \), the web \( W(n, p) \) has \( n \) vertices and edges \( E = \{(i, j) : j = i+p, \ldots, i+n-p; \ \forall \ \text{vertices } i\} \)
Web Inequalities

• Thm (Trotter): If \( \gcd(n, p) = 1 \), then 
  \[ \sum_{v \in V(W(n,p))} x_v \leq p \] 
is a facet for stable set polytope of \( W(n, p) \).

• Thm (McClosky & Hicks): If \( \gcd(n, p + 1) = 1 \), then 
  \[ \sum_{v \in V(W(n,p))} x_v \leq p + 1 \] 
is a facet for co-2-plex polytope of \( W(n, p) \).
$k$-claw-free graphs

- claw-free graphs
  - Combinatorial algorithm (Minty 1980)
  - Full polyhedral structure unknown

- 2-claw-free graphs
  - Full polyhedral structure (McClosky & Hicks 2009)
Given an integer \( k \geq 1 \), the graph \( G \) is a \( k \)-claw if there exists a vertex \( v \) of \( G \) whose neighbors (at least \( \max\{3, k\} \)) make up the rest of the graph and form a co-\( k \)-plex.
A clutter is a pair \((V, E)\) where \(V\) is a finite set and \(E\) is a family of subsets of \(V\) none of which is included in another.
Clutters of Maximal 2-plexes

- Given a graph $G$, let $C$ be the clutter whose vertices are $V(G)$ and whose edges are maximal 2-plexes of $G$. 

![Graph G and Clutter C](image)
Clutters of Maximal 2-plexes
2-claw free & 2-plex Clutter Matrices

- Thm (McCloksy & Hicks 2009): A graph G is 2-claw free if and only if $\Delta(G) \leq 2$ or G is 2-plex.

- Thm (McClosky & Hicks, 2009): Let $A$ be the 2-plex clutter matrix of G. The polytope $P=\{x \in \mathbb{R}^n_+: Ax \leq 2, x \leq 1\}$ is integral if and only if the components of G are 2-plexes, co-2-plexes, paths, or 0 mod 3 chordless cycles.

- Corollary (McClosky & Hicks, 2009): Given a 2-plex clutter matrix $A$, there is a polynomial-time algorithm to determine if $P=\{x \in \mathbb{R}^n_+: Ax \leq 2, x \leq 1\}$ is integral.
Analogs in Polyhedral Combinatorics

- K-plex Polytope

- Co-2-plex Polytope

- 2-Club Polytope (Pajouh, Balasundaram, Hicks, 2016)
Analogs in Polyhedral Combinatorics for 2-club polytope

- Padberg (1973)
  - odd hole inequalities
  - Stable set inequalities
Thm: If $C$ is an $n$-cycle such that $n > 6$ and $n \neq 0$ mod 3, then 
\[ \sum_{v \in V(C)} x_v \leq 3 \] 
is a facet for 2-club polytope.
Stable Sets and Dominating Sets

• Given a graph $G=(V, E)$, a set $D \subseteq V$ is called a **dominating set** if for every node $v \in V \setminus D$ there exists $w \in D$ such that $vw$ is an edge ($d(v, w) = 1$).

• A dominating set that is also a stable set is called an **independent dominating set**.

![Graph Diagram]

1  2  3  4  5  6
• Theorem (Padberg 1973): If $S$ is a maximal stable set in $G$, then $\sum_{v \in S} x_v \leq 1$ is a facet for $P_1(G)$.

• Recall, Berge observed that a stable set is maximal if and only if it is dominating.
Given a graph \( G=(V, E) \) and some integer \( k > 0 \), a set \( D \subseteq V \) is called a \( k \)-dominating set if for every node \( v \in V \setminus D \) there exists \( w \in D \) such that \( d(v, w) \leq k \).

A \( k \)-dominating set that is also independent is called an independent \( k \)-dominating set.
2-club Polytope: I2D inequalities

- Notation: $k^+ = \max\{k, 0\}$
- Theorem: Given a graph $G=(V, E)$ and an independent 2-dominating (I2D) set $C \subseteq V$, then
  \[ x(C) - \sum_{v \in V \setminus C} (|N(v) \cap C| - 1)^+ x_v \leq 1 \] is facet inducing for $P_2(G)$.

\[ -x_1 + x_2 - x_3 - x_4 + x_5 + x_6 \leq 1 \]
\[ -x_4 + x_5 + x_6 \leq 1 \]
\[ x_3 \leq 1 \]
I2D Facets

• Theorem: These I2D facets fully characterize 2-club polytope for acyclic graphs.

• All previous known non-trivial facets for the polytope are special cases of I2D inequalities
  – 2 independent sets (independent by a distance of 2) (Balasundaram et al. (2005))
  – Pair-blob (Carvalho et al. (2008))
  – Roof-blob (Carvalho et al. (2008))

• The separation problem for I2Ds is NP-hard.
Wrap-Up

• Social Networks

• $k$-plexes & co-$k$-plexes & $k$-clubs

• Combinatorial algorithm analogs

• Polyhedral Combinatorics analogs
Any Questions?
Time to Go