Discrete Optimization and Network Analysis

Illya V. Hicks
Computational and Applied Mathematics
Rice University

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Butenko

Collaborators

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- Foad Pajouh (Umass-Boston)
- Cynthia Wood (SAP)

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Our Helpers For Today

I Can Tell You My Secret Now?

I see graphs everywhere!

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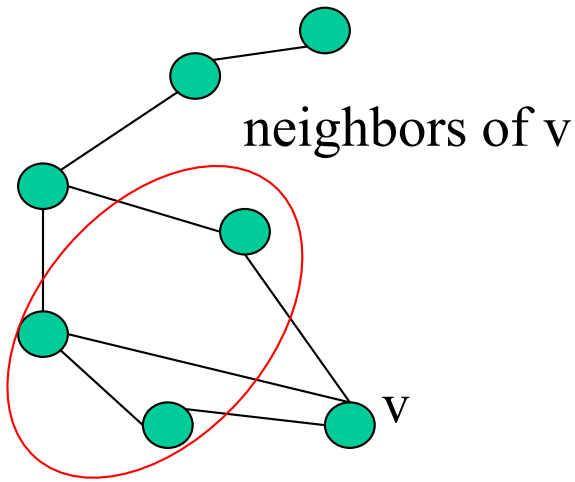
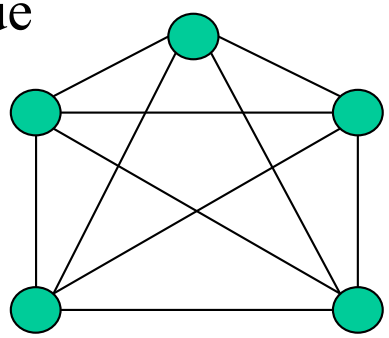
Graphs (Networks)

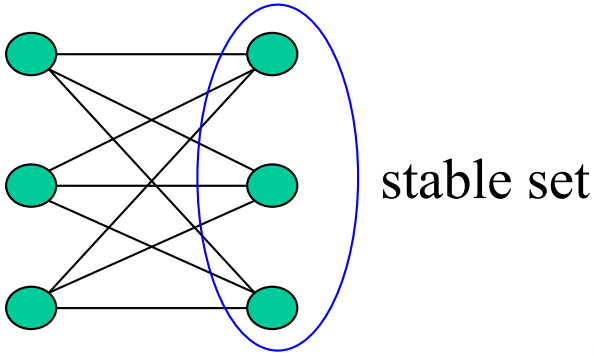
Graph G=(V, E)

- Vertex set V is finite
- Edges E = {uv : u,v ∈ V}
- Undirected (for this talk)
- u is a neighbor of w if uw ∈ E

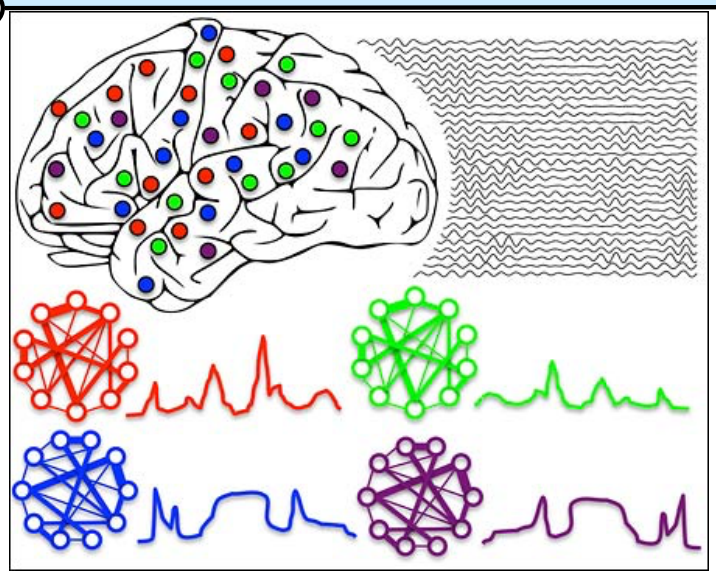


clique

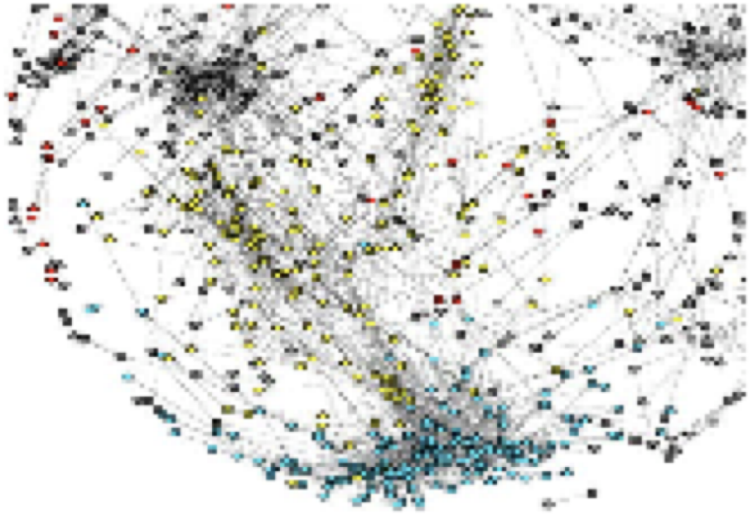




Networks are Everywhere!



Yang 2010



Carlson et al. 2006

Don Johnson, Rice

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Who Likes Bacon?

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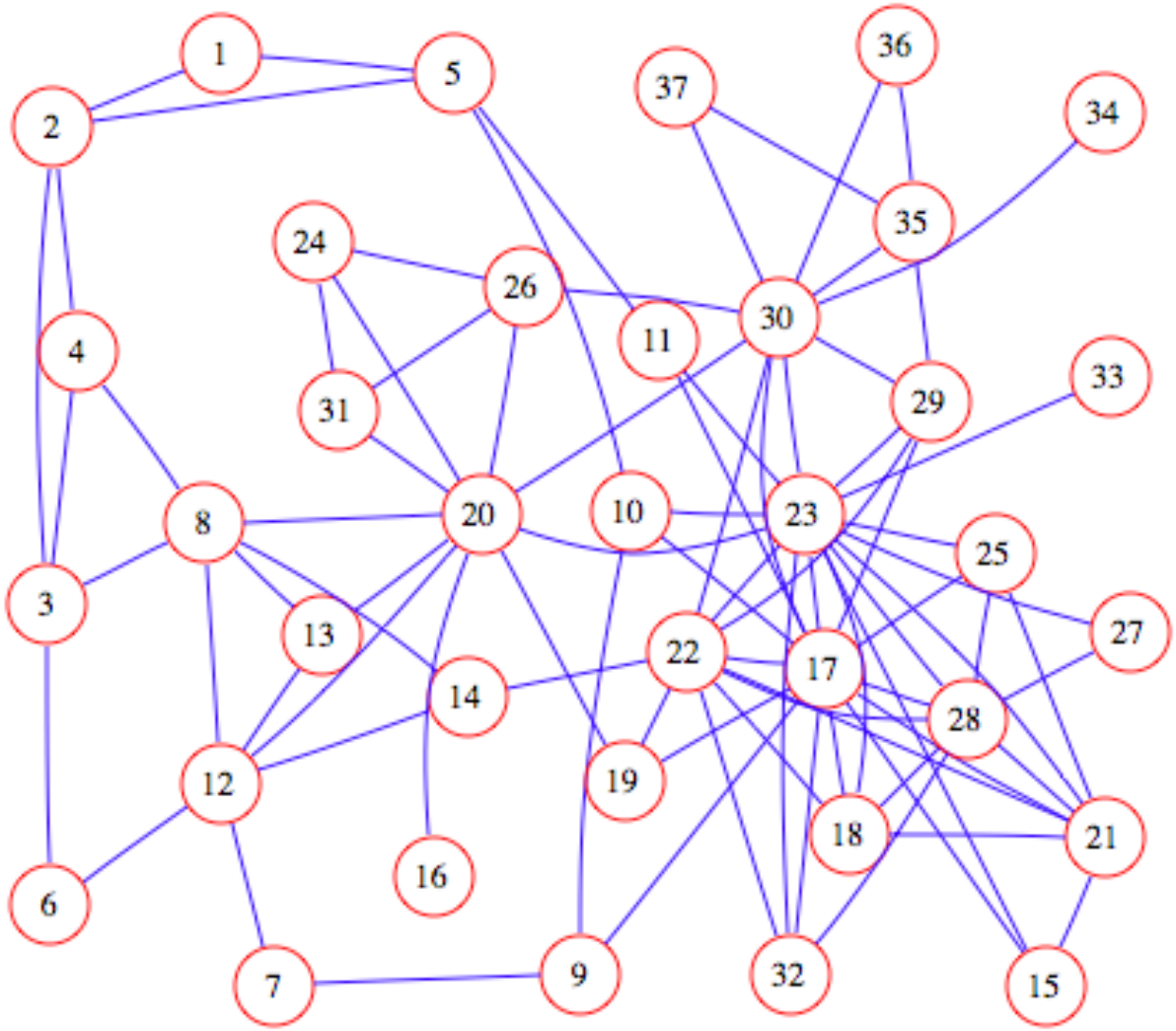
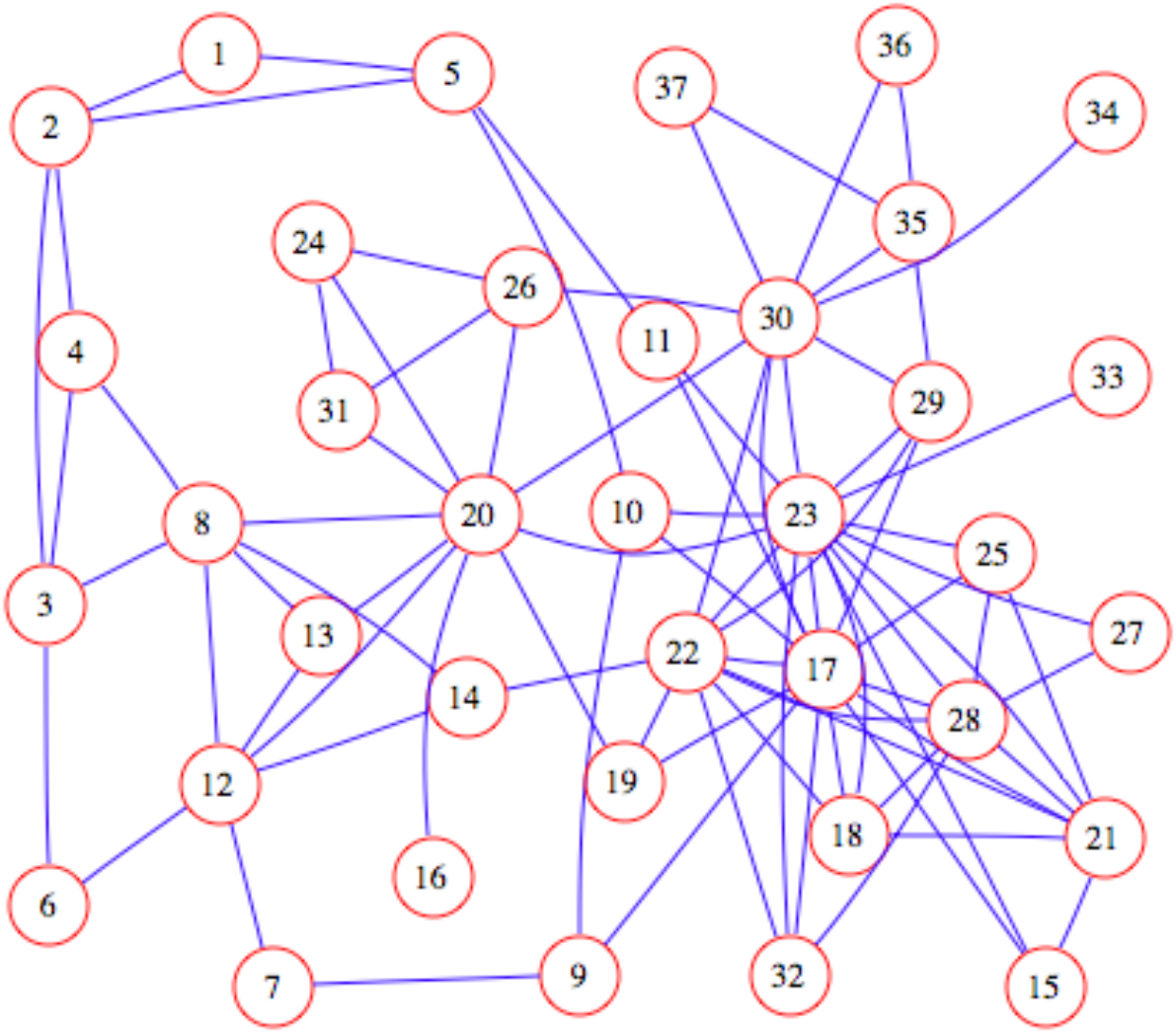
Social Network Pop Quiz



9-11Terrorist Network

1) Alshehri
2) Sugami
3) Al-Marabh
4) Hijazi
5) W. Alshehri
6) A. Alghamdi
7) M. Alshehri
8) S. Alghamdi
9) Ahmed
10) Al-Hisawi
11) Al-Omari
12) H. Alghamdi
13) Alnami
14) Al-Haznawi
15) Darkazanli
16) Abdi
17) Al-Shehhi
18) Essabar
19) S. Alhazmi



20) N. Alhazmi
21) Bahaji
22) Jarrah
23) Atta
24) Shaikh
25) El Motassadeq
26) Al-Mihdhar
27) Moussaoui
28) Al-Shibh
29) Raissi
30) Hanjour
31) Awadallah
32) Budiman
33) Al-ani
34) Moqed
35) Abdullah
36) Al Salmi
37) Alhazmi

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Homer Ignoring Lisa

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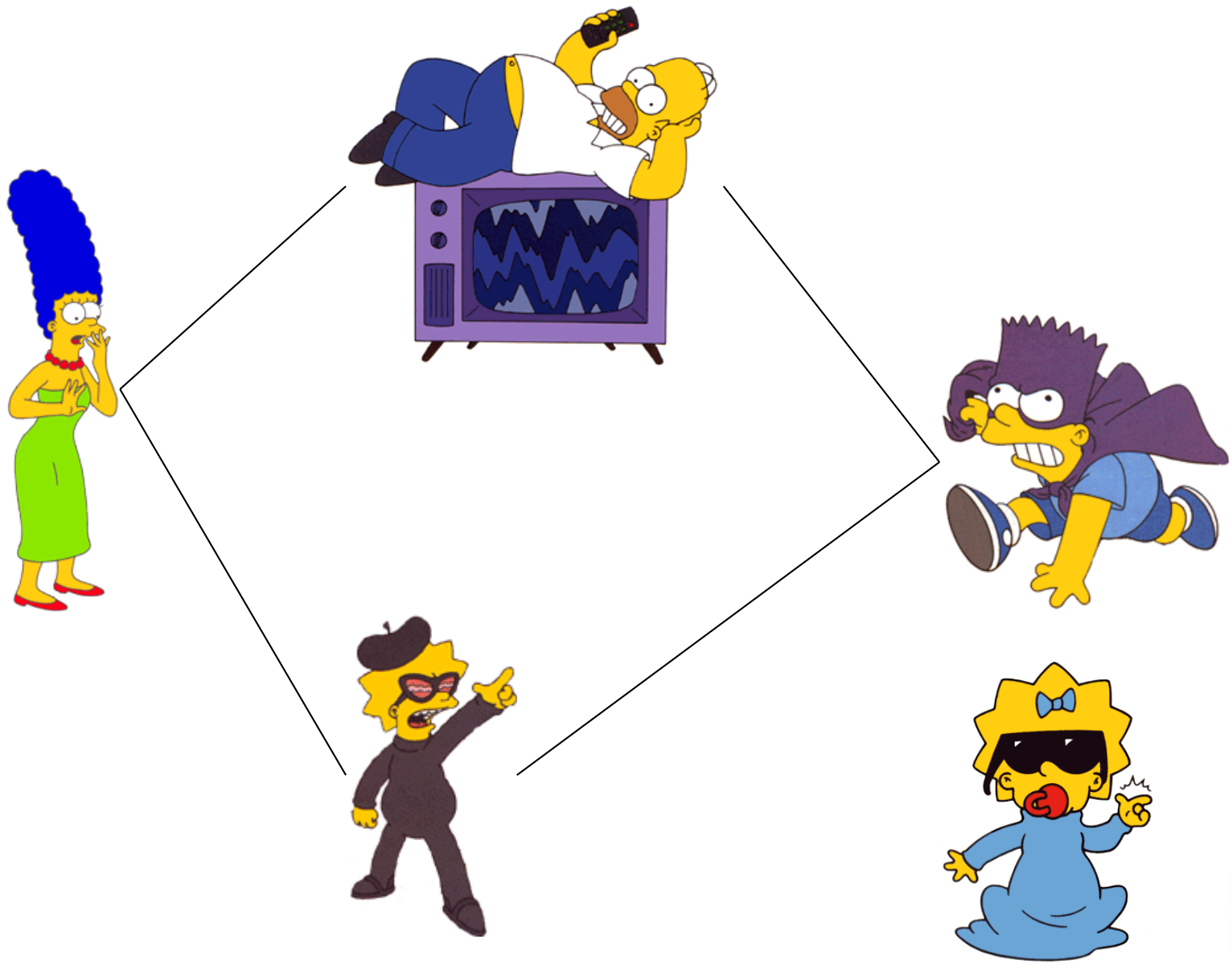
The Simpsons Social Network



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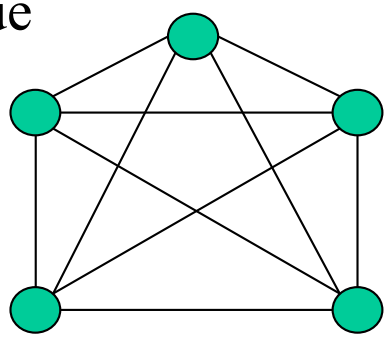
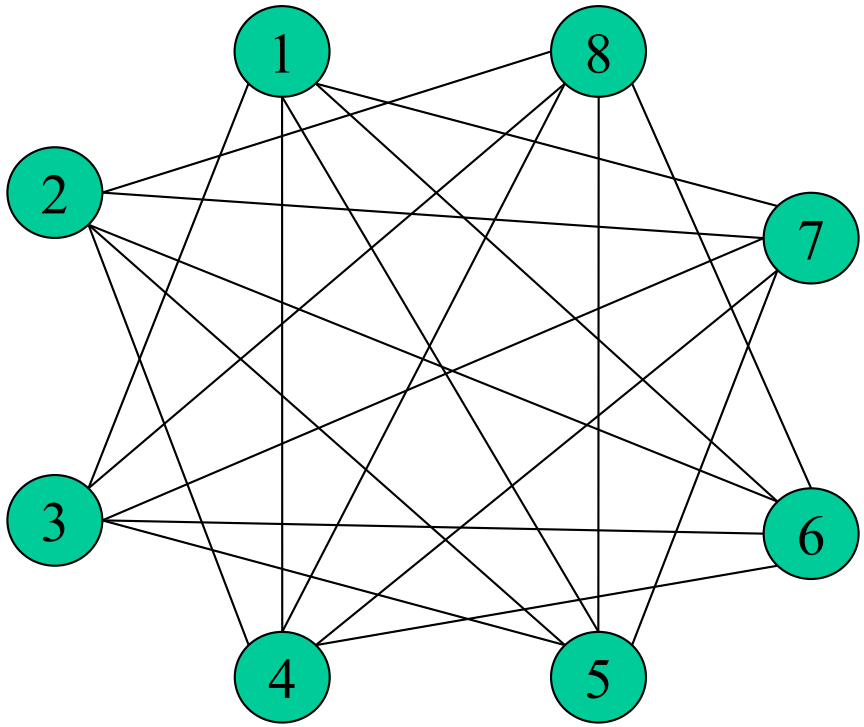
What is cohesiveness in terms of graphs?

clique



- Debated by social scientists
- Three general properties
  - Familiarity (few strangers)
  - Members can easily reach each other (quick communication)
  - Robustness (not easily destroyed by removing members)

Is this graph cohesive?



Clique is too restrictive!

Different versions of cohesiveness

- Relax distance requirement between members
  - *k*-clique (Luce 1950)
  - *k*-club (Alba 1973)
- Relax the familiarity (# of neighbors) between members
  - *k*-plex (Siedman & Foster 1978)
  - *k*-core (Siedman 1983)

Others: F-group, LS sets, Lambda sets, quasi-cliques, k-trusses

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Talks at this Conference on Clique Generalizations

- "An Upper-bounding Technique for the Maximum Quasi-clique Problem" by Balasundaram and Miao
- "Algorithms for the Maximum Edge Weight Clique Problem" by Fontes, Butenko, and Hosseinian
- "Parsimonious Formulations for Low-Diameter Clusters" by Salemi and Buchanan

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Clarity

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Outline

I. Definitions

II. My Research in the Area

III. Conclusions

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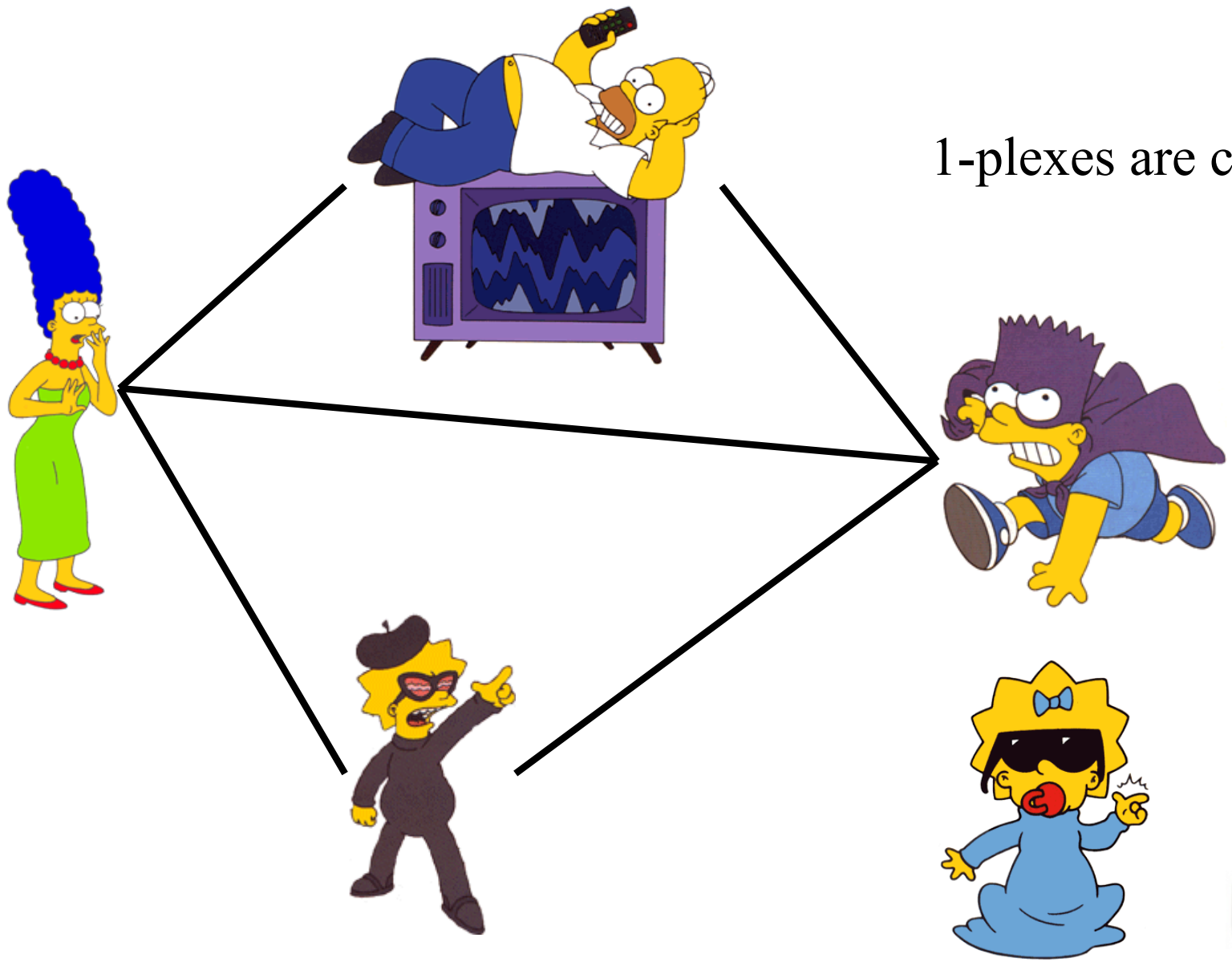
k-plexes

- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a k-plex if every node of S has at most k-1 non-neighbors in S

- Cliques are 1-plexes

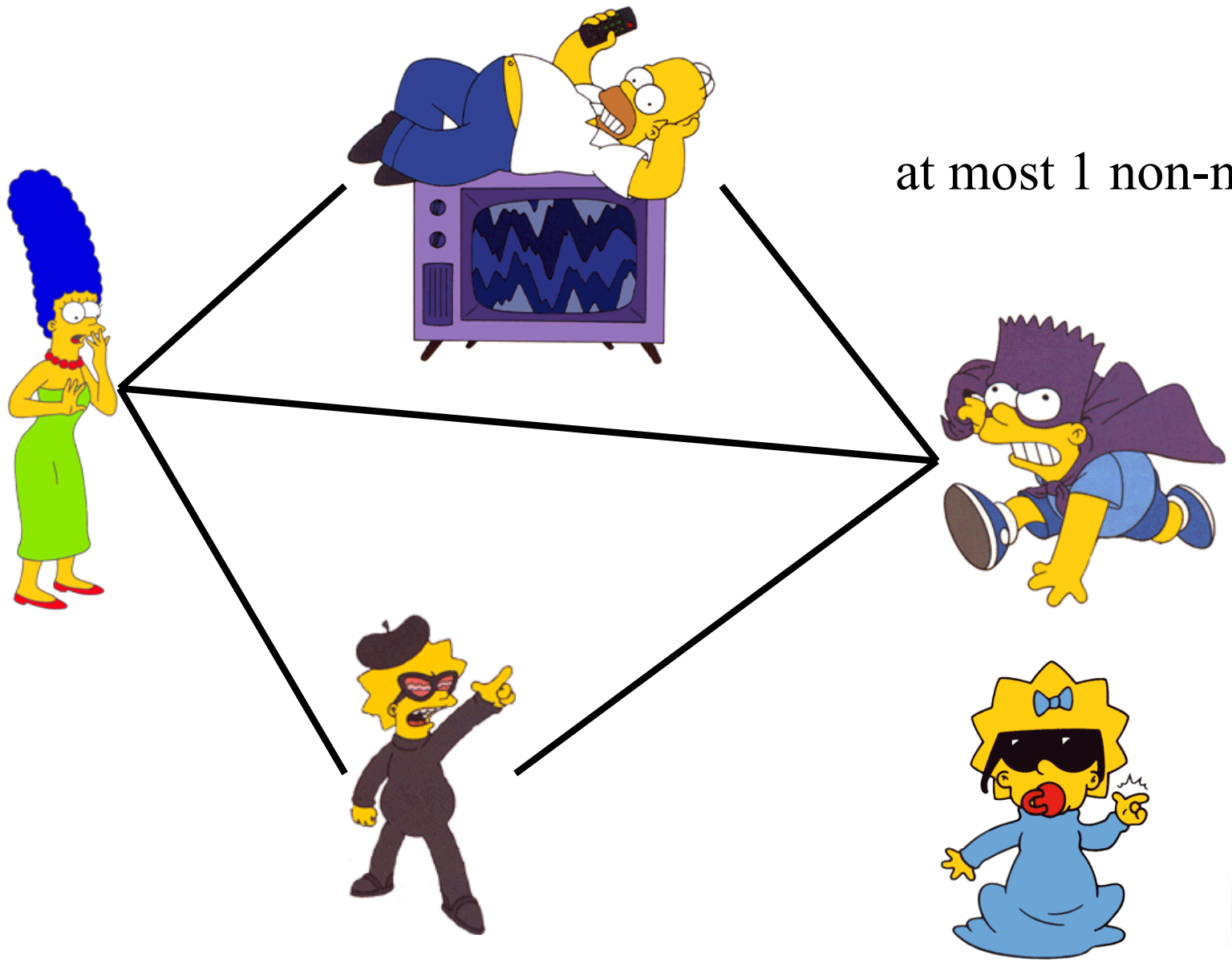
- NP-hard to find maximum k-plex, $\omega_k(G)$, in a graph G

1-plexes



1-plexes are cliques

2-plexes



at most 1 non-neighbor

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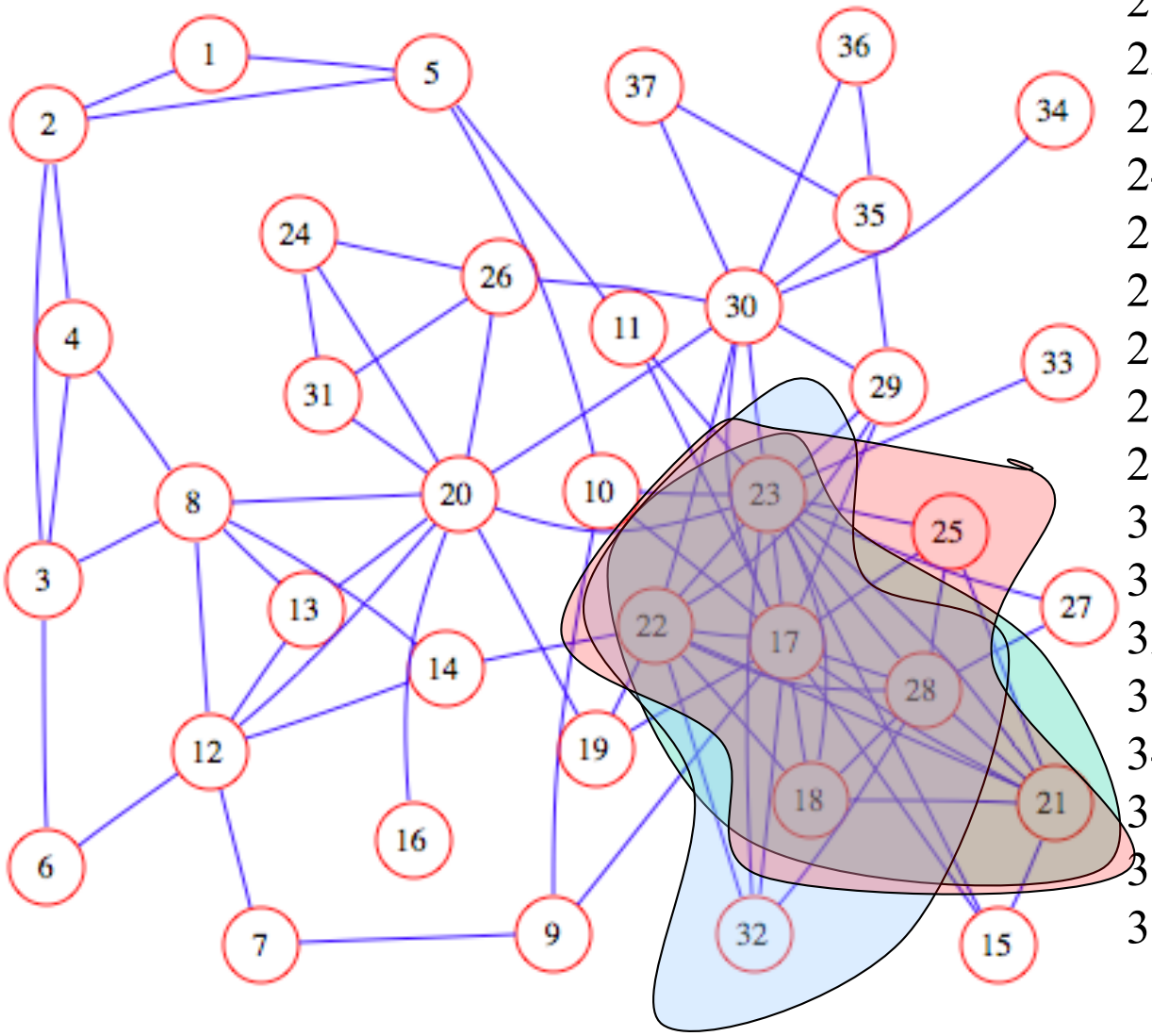

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37) Alhazmi

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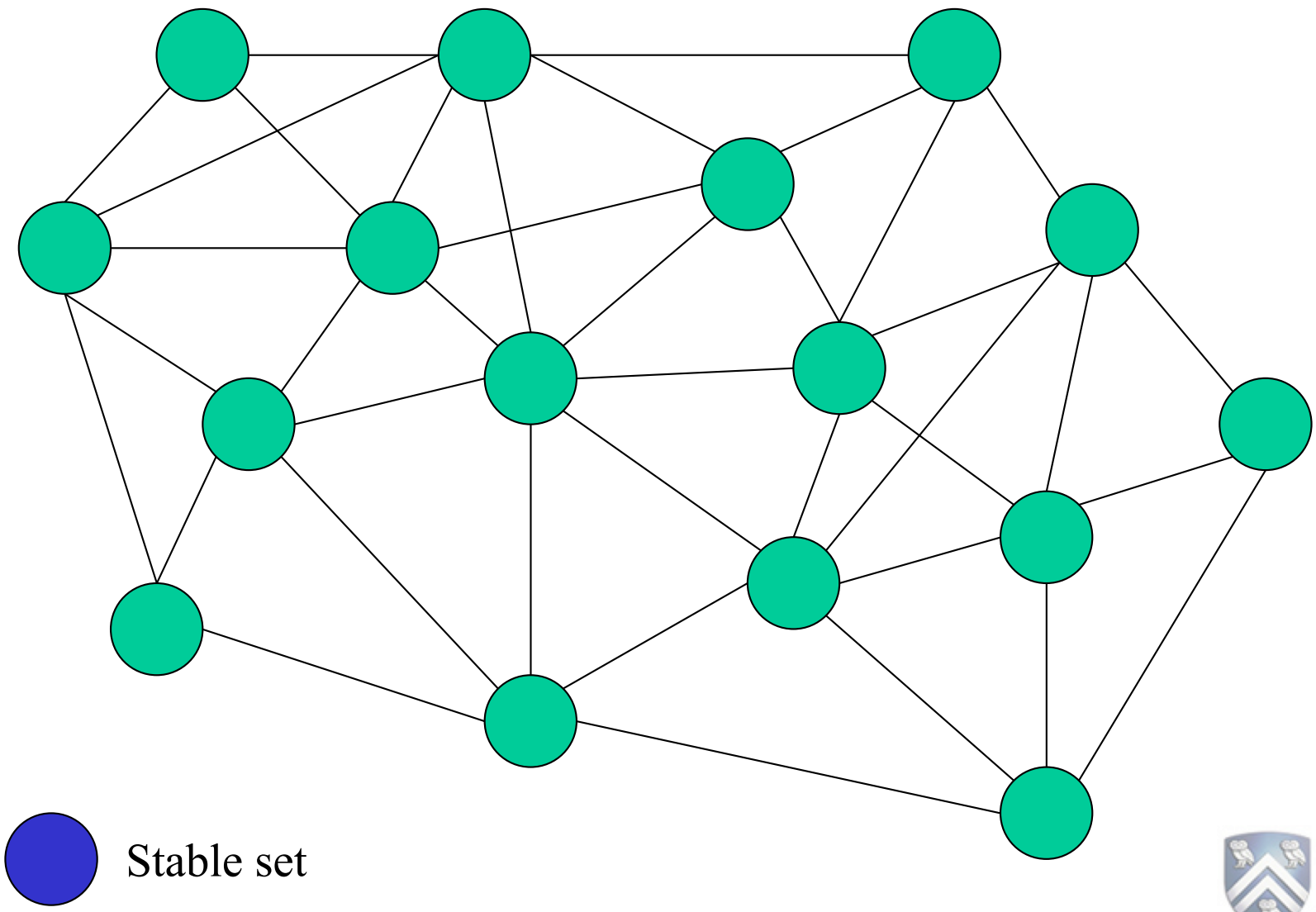
Another Example: Retail Location



Stable set

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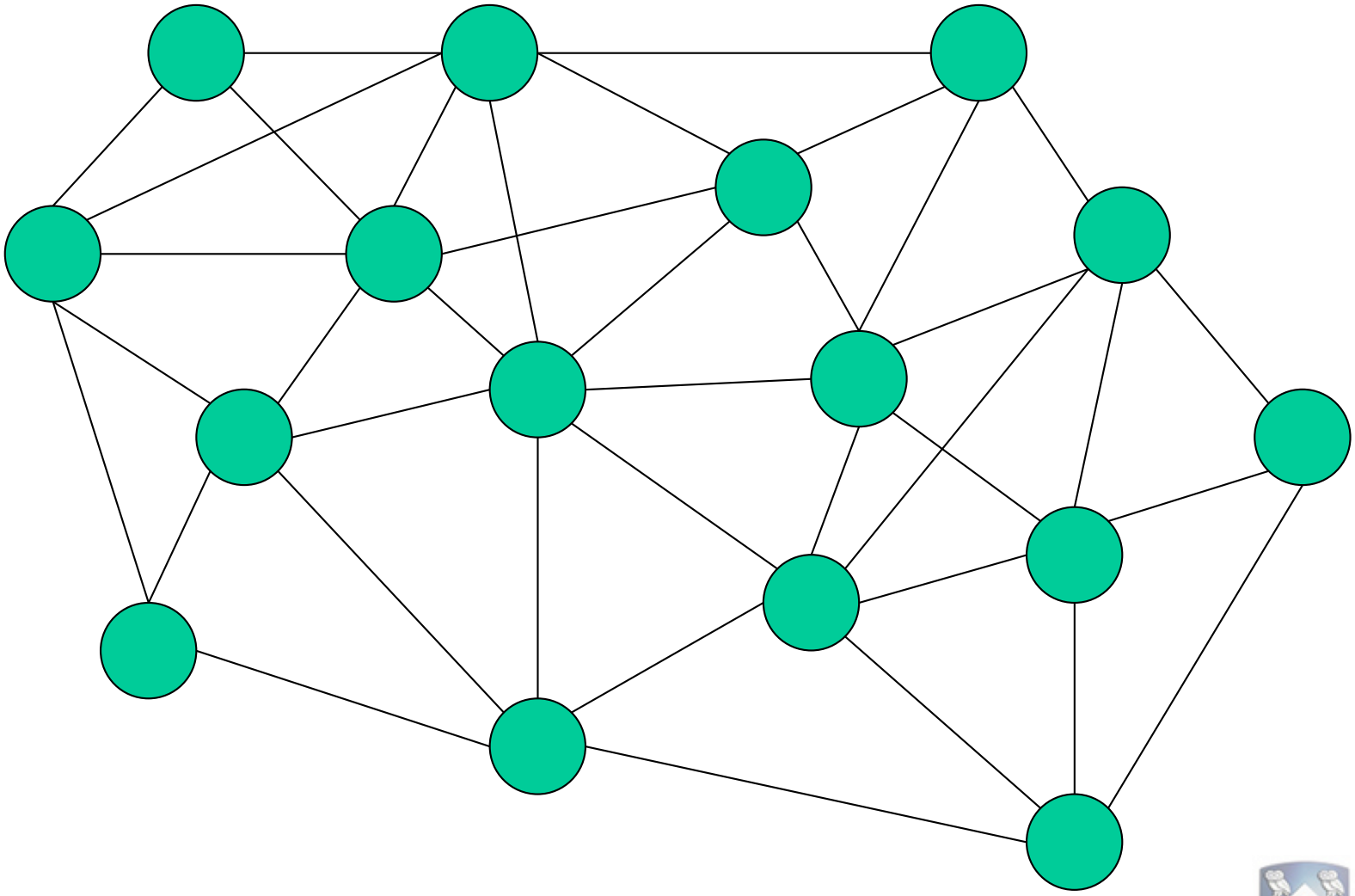
Starbucks in Springfield

Another Example: Retail Location



Co-k-plexes

- A set $S \subseteq V$ is called a co-k-plex if every node of S has at most k-1 neighbors in S
- Stable sets are co-1-plexes
- NP-hard to find maximum co-k-plex, $\alpha_k(G)$ in a graph G
- Co-2-plexes correspond to vertex induced subgraphs of isolated nodes and matched pairs

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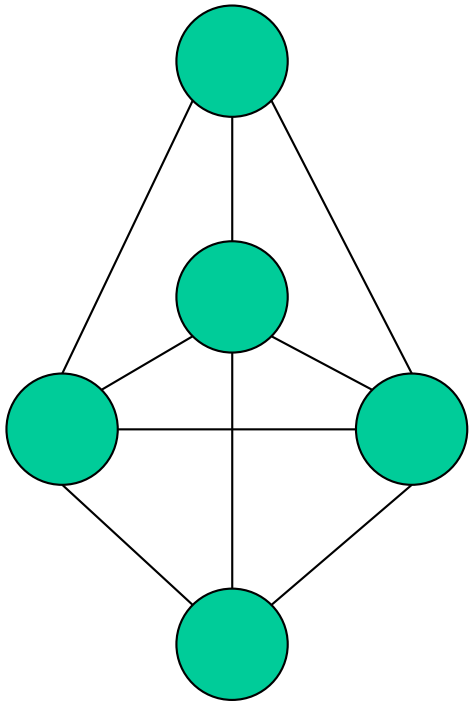
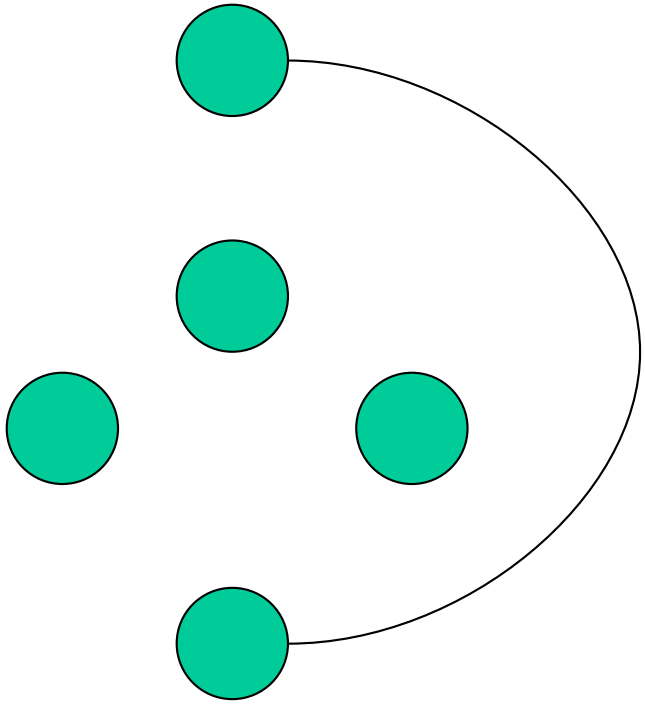
k-plexes and co-k-plexes are equivalent problems



G



Complement of G

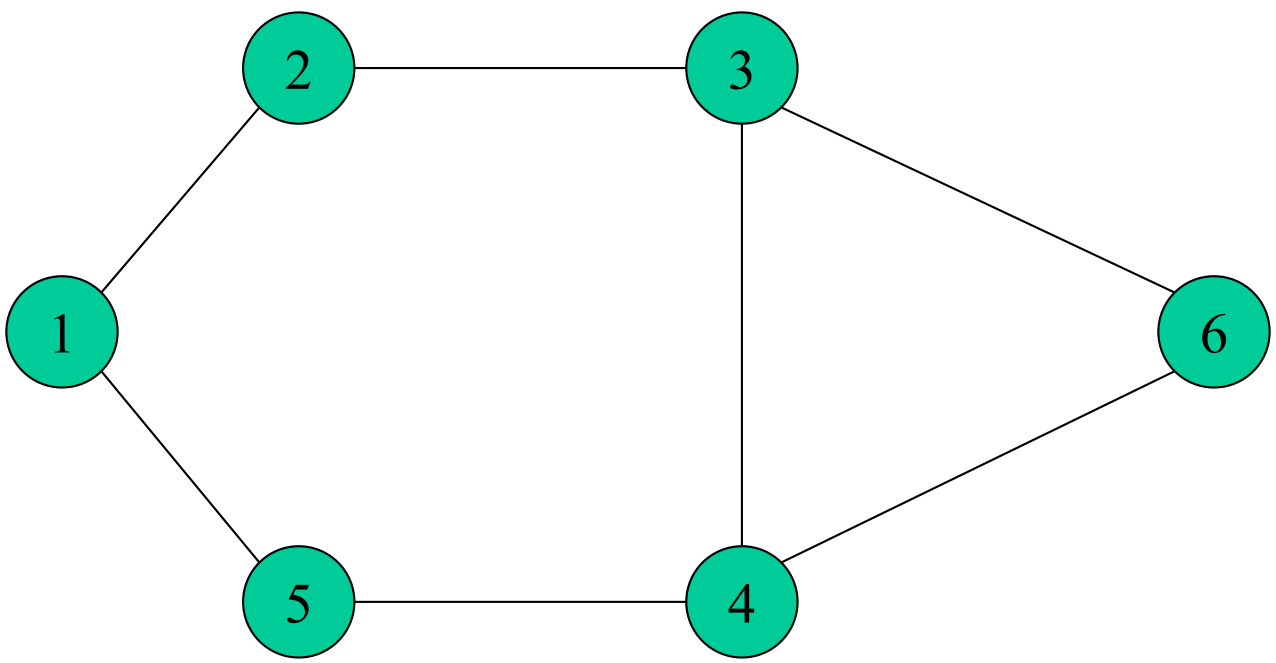
Not k-clubbing!

k-clubs

- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a k-club if $\text{diam}(G[S]) \leq k$

2-club

4-club

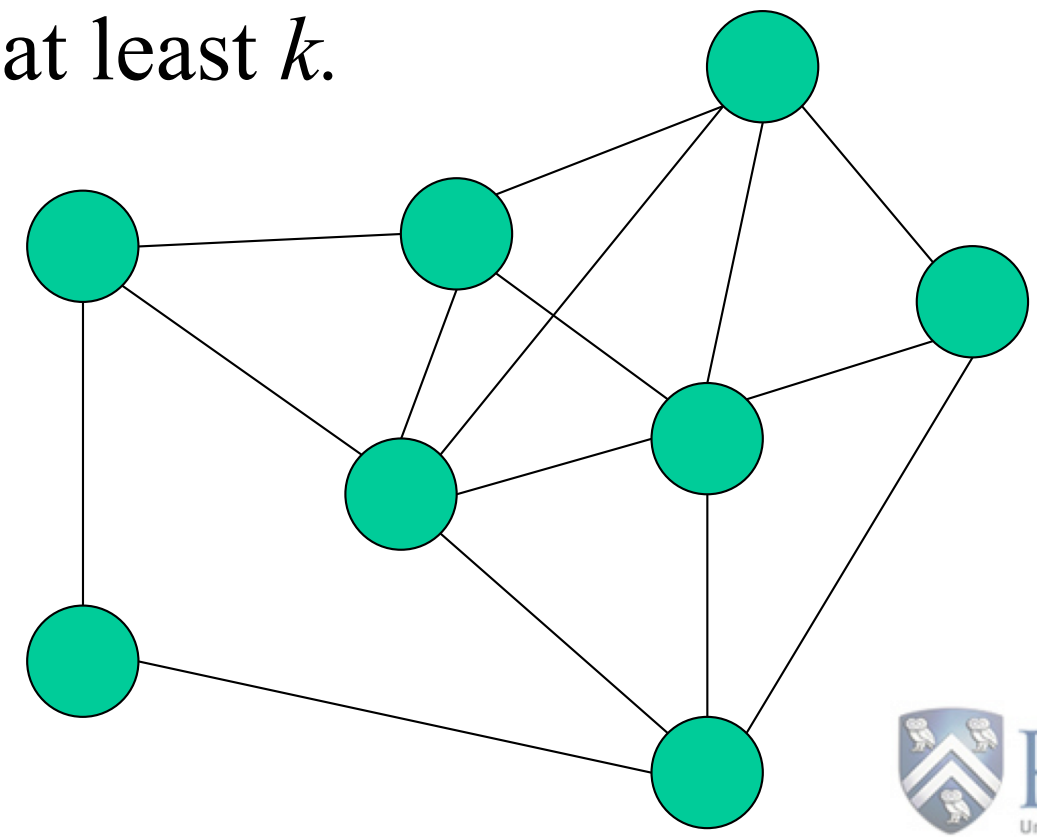


K-cores

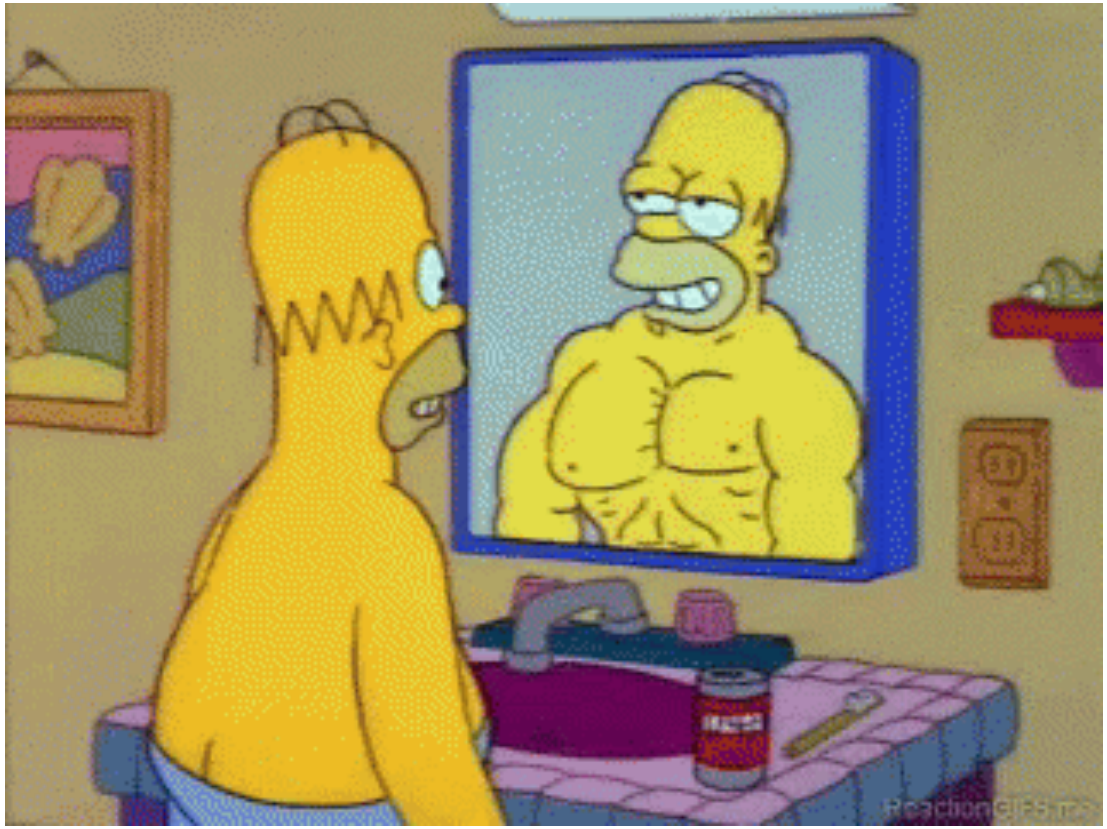
- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a *k*-core if every node of G[S] has degree at least k.

3-core

NP-hard for minimum k-core



My Research: Analogs



Analogs in Combinatorial Algorithms
Analogs in Polyhedral Combinatorics

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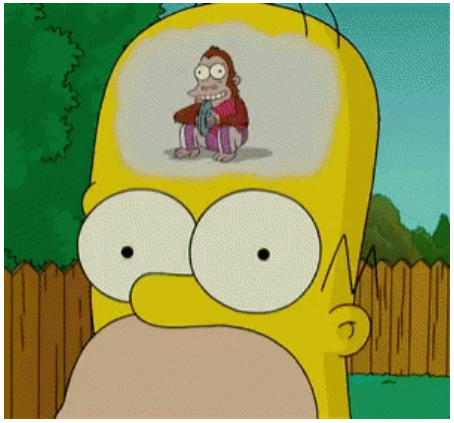
Analogs in Combinatorial Algorithms

- K-plexes (McClosky & Hicks, 2012)
- Co-plex coloring (McClosky & Hicks 2012; McClosky, Arellano & Hicks, 2015)
- K-cores (Wood & Hicks, 2015)
- Co-2-plexes in {claw,bull}-free graphs (Wood, McClosky & Hicks)



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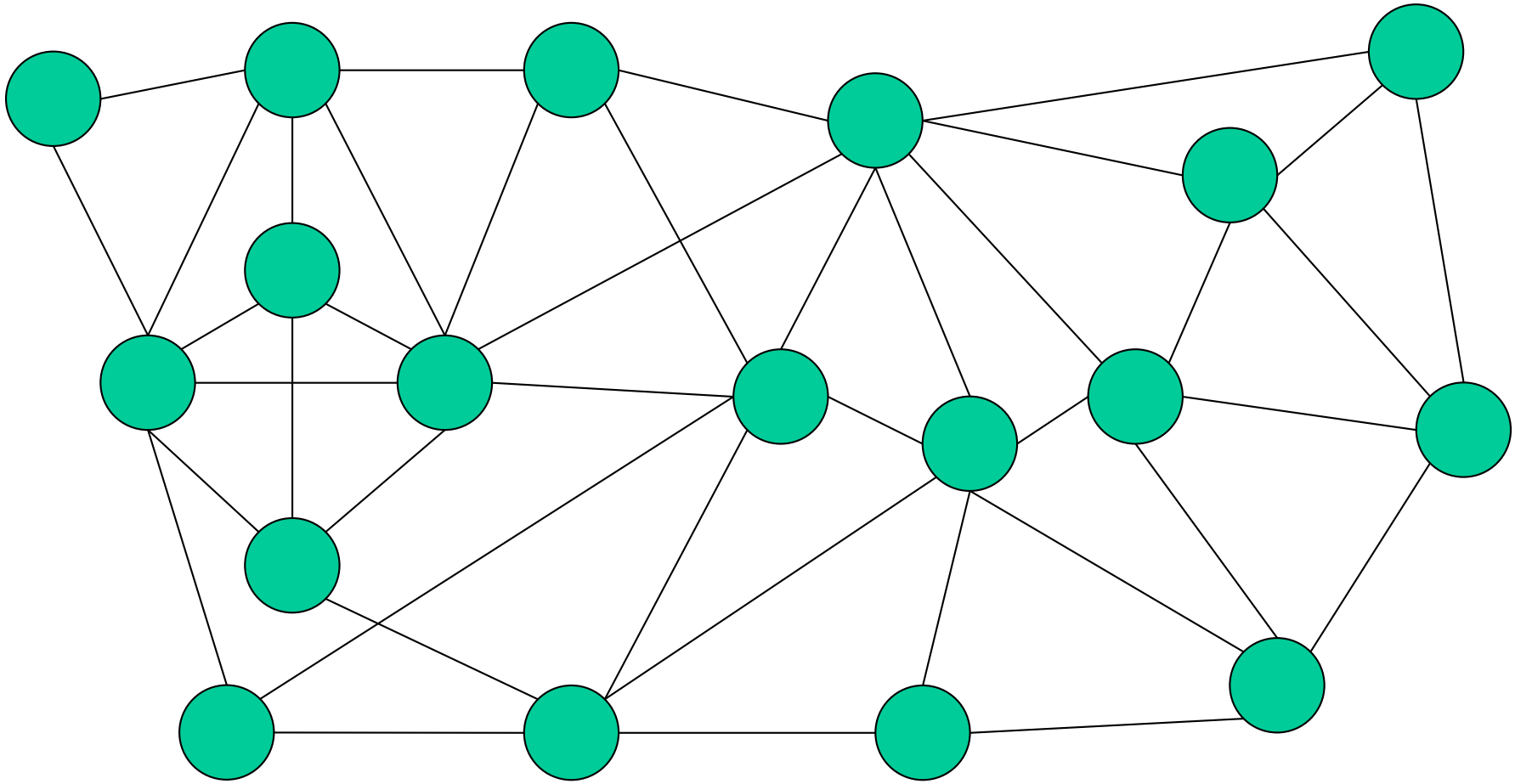
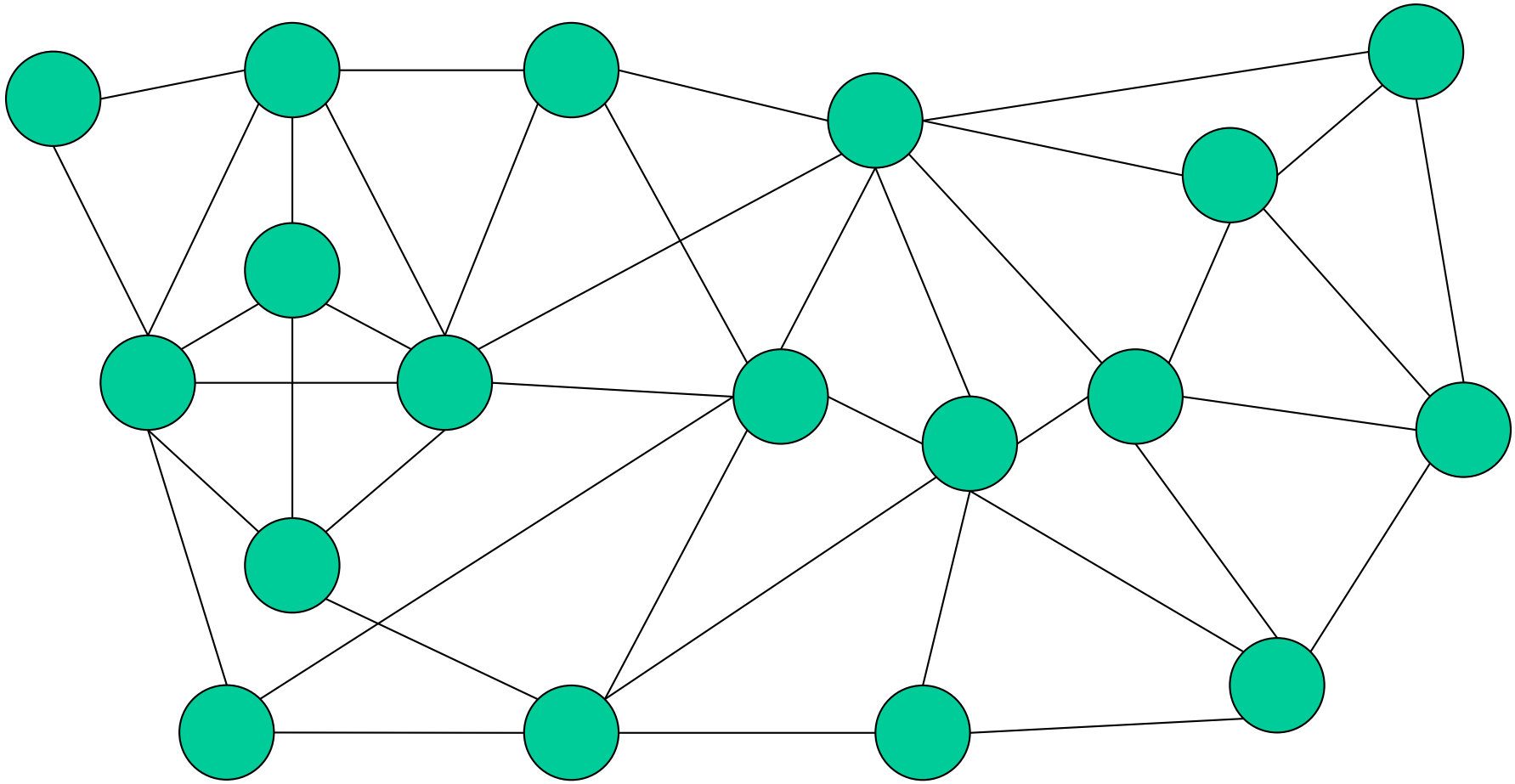
Graph Coloring



Clique number $\omega(G) \leq \chi(G)$ Chromatic number

Co-k-plex Coloring



K-plex number $\omega_k(G) \leq \chi_k(G)$ Co-k-plex chromatic number

McClosky & Hicks (2012)

Co-k-plex Coloring Properties

- Thm (McClosky, Arellano & Hicks, 2015): For any graph G, $\chi_k(G) \leq k\chi(G)$.
- $M = (S, I)$ with finite set S and family I of subsets of S is called a *matroid* if the following axioms are satisfied
  - $\varnothing \in I$
  - if $J' \subseteq J \in I$, then $J' \in I$
  - for every $A \subseteq S$, every maximal independent subset of A has the same cardinality

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Co-k-plex Coloring Properties

- A graph is k-plex matroidal if the (V(G), K) is a matroid where K is the family of vertex subsets of G that form k-plexes.

- Thm (McClosky, Arellano & Hicks, 2015): If graph G is k-plex matroidal then $\omega_k(G') = \chi_k(G')$ for all vertex induced subgraphs G'

- Also gave analog of Kubale & Jackowski (1985)

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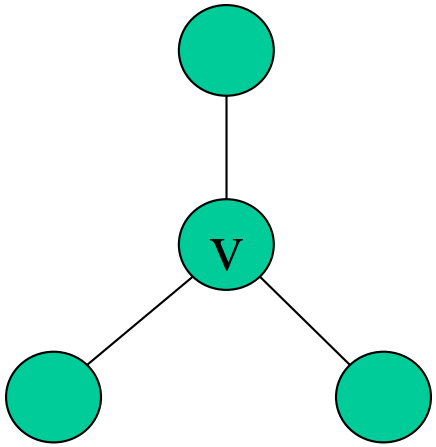
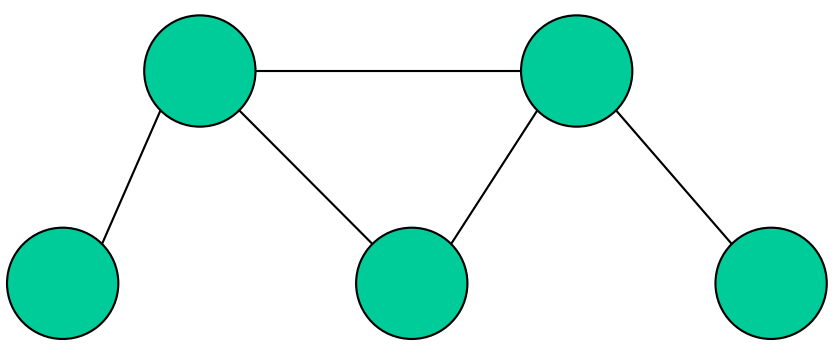
Claw-free graphs

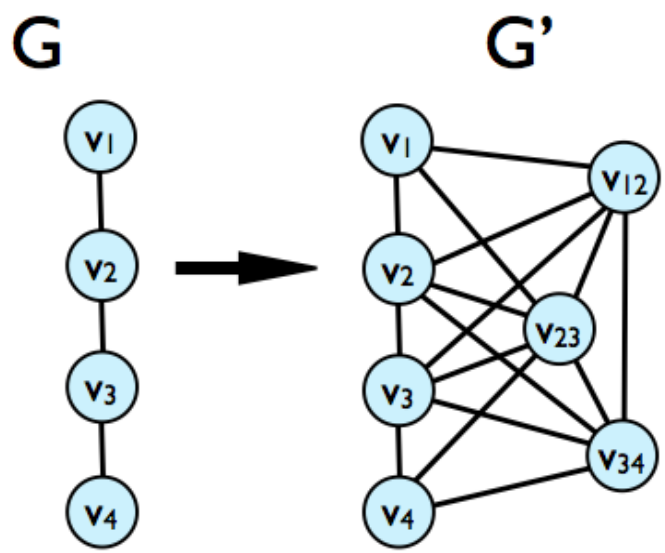
- Minty (1980)
  - Combinatorial algorithm for claw free graphs



{claw,bull}-free graphs

- Two approaches
  - Transformation from {claw, bull}-free to claw-free
  - Direct algorithm
  - Both polynomial-time combinatorial algorithms





Analogs in Polyhedral Combinatorics

- K-plex Polytope
- Co-2-plex Polytope
- 2-Club Polytope

Polyhedral Combinatorics Analogs for general k

- K-plex polytope (Balasundaram, Butenko & Hicks, 2011)
  - Maximal stable sets
  - Holes (certain types)

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Analogs in Polyhedral Combinatorics for co-2-plex polytope

- Padberg (1973)
  - Clique and odd hole inequalities
- Trotter (1975)
  - Web inequalities
- Minty (1980)
  - Claw free graphs

2-plex Inequalities

- Thm (Padberg, 1973): If K is a maximal clique in G, then $\sum_{v \in K} x_v \leq 1$ is a facet for stable set polytope of G.

- Thm [(McClosky & Hicks, 2009), (Balasundaram et al., 2011)]: If K is a maximal 2-plex in G such that $|K| > 2$, then $\sum_{v \in K} x_v \leq 2$ is a facet for co-2-plex polytope of G.

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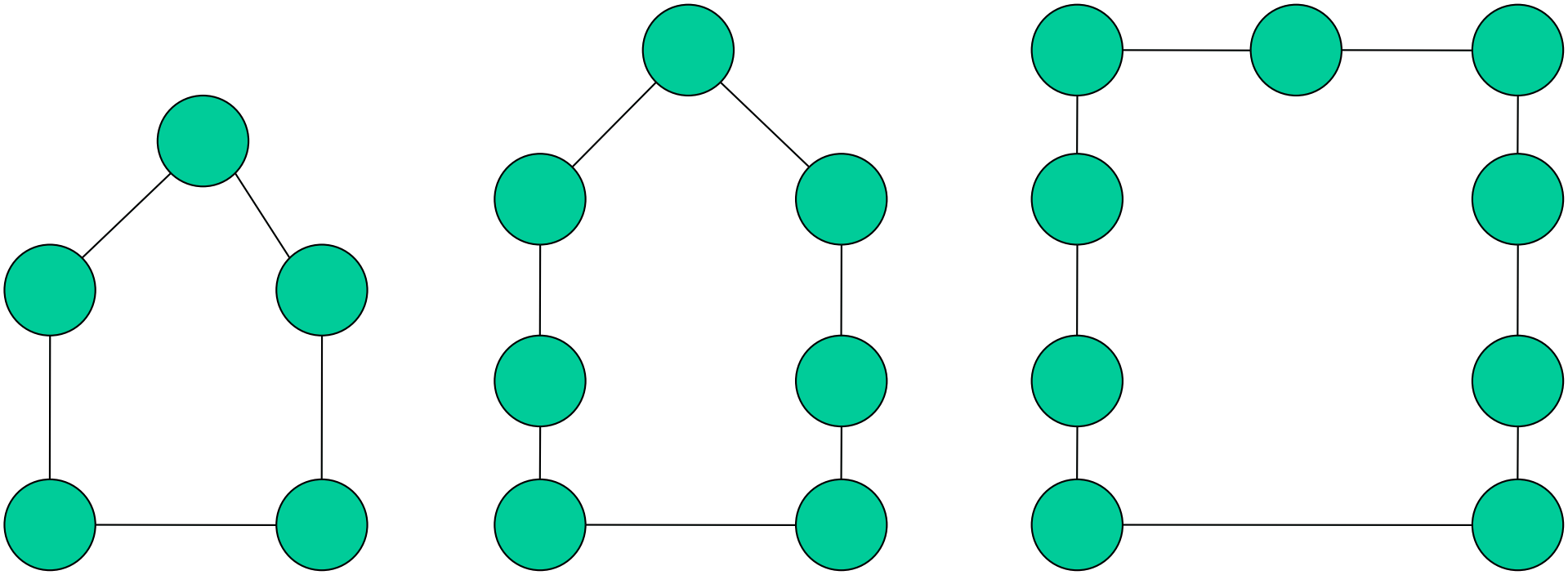
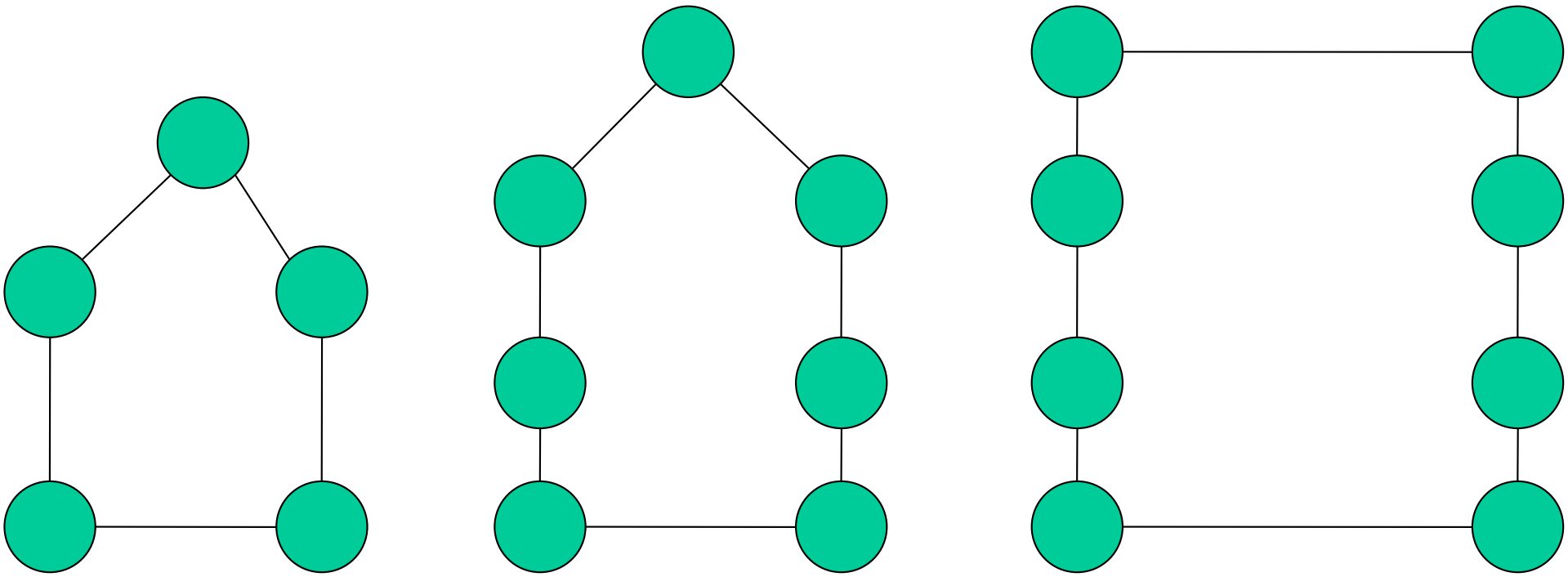
Odd Holes



Thm (Padberg, 1973): If C is an n-chordless cycle such that $n > 3$ is odd, then $\sum_{v \in V(C)} x_v \leq \lfloor n/2 \rfloor$ is a facet for the stable set polytope.

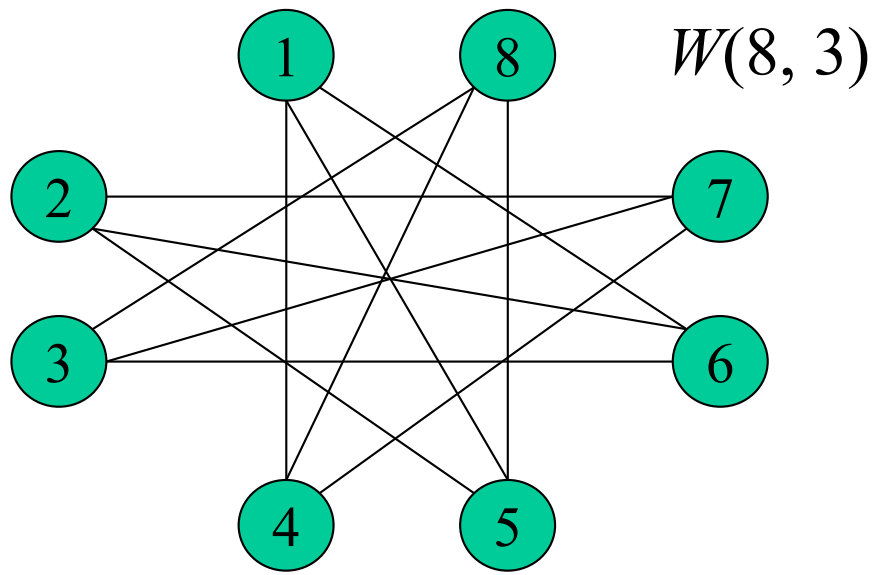
Odd Mod 3 Holes



Thm (McClosky & Hicks, 2009): If C is an n-chordless cycle such that $n > 2$ and $n \neq 0 \bmod 3$, then $\sum_{v \in V(C)} x_v \leq \lfloor 2n/3 \rfloor$ is a facet for the co-2-plex polytope.

Webs

- For fixed integers $n \geq 1$ and p such that $1 \leq p \leq \lfloor n/2 \rfloor$, the web $W(n, p)$ has n vertices and edges $E=\{(i, j): j=i+p, \ldots, i+n-p; \forall$ vertices $i\}$



Web Inequalities

- Thm (Trotter): If $\gcd(n, p) = 1$, then $\sum_{v \in V(W(n,p))} x_v \leq p$ is a facet for stable set polytope of $W(n, p)$.

- Thm (McClosky & Hicks): If $\gcd(n, p + 1) = 1$, then $\sum_{v \in V(W(n,p))} x_v \leq p + 1$ is a facet for co-2-plex polytope of $W(n, p)$.

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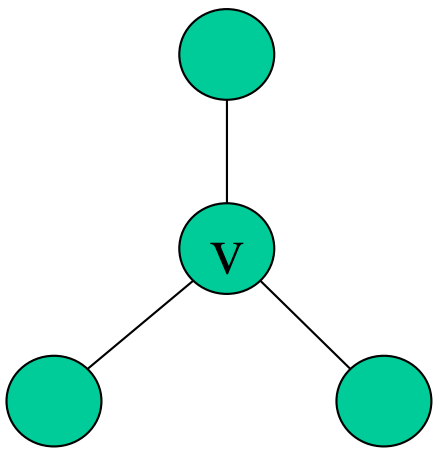
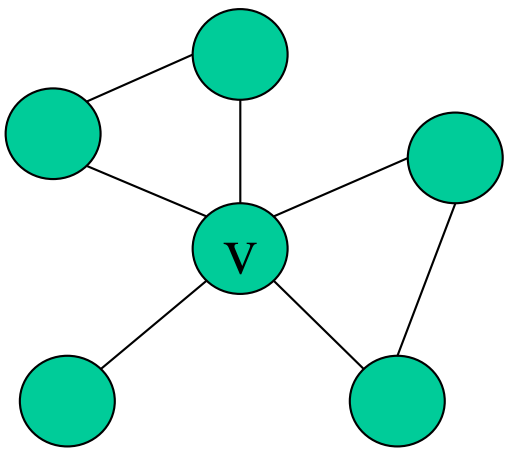
k-claw-free graphs

- claw-free graphs
  - Combinatorial algorithm (Minty 1980)
  - Full polyhedral structure unknown
- 2-claw-free graphs
  - Full polyhedral structure (McClosky & Hicks 2009)

k-claws

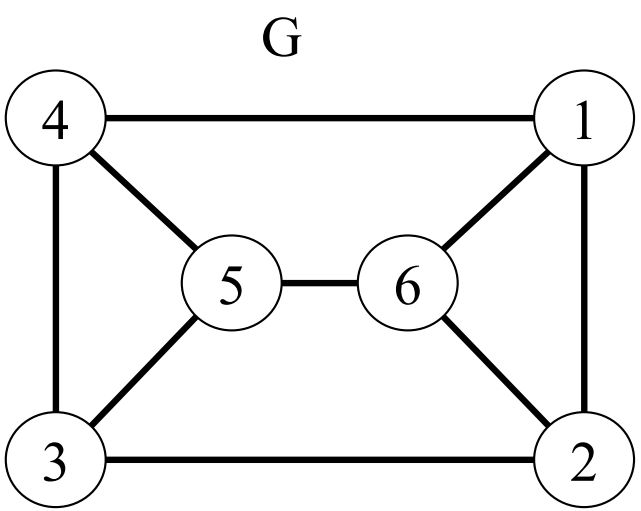
- Given an integer $k \geq 1$, the graph G is a k-claw if there exists a vertex v of G whose neighbors (at least $\max\{3, k\}$) make up the rest of the graph and form a co-k-plex.

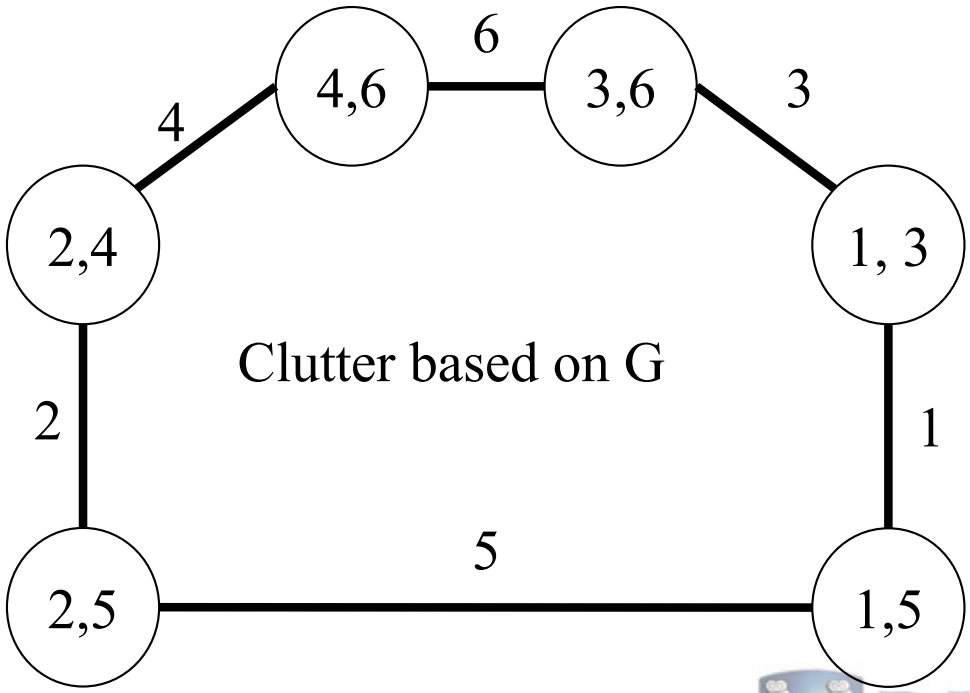




Clutters

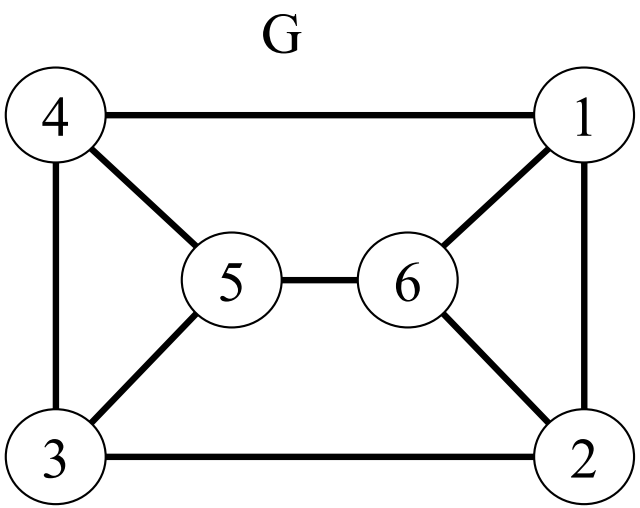
- A clutter is a pair (V, E) where V is a finite set and E is a family of subsets of V none of which is included in another.

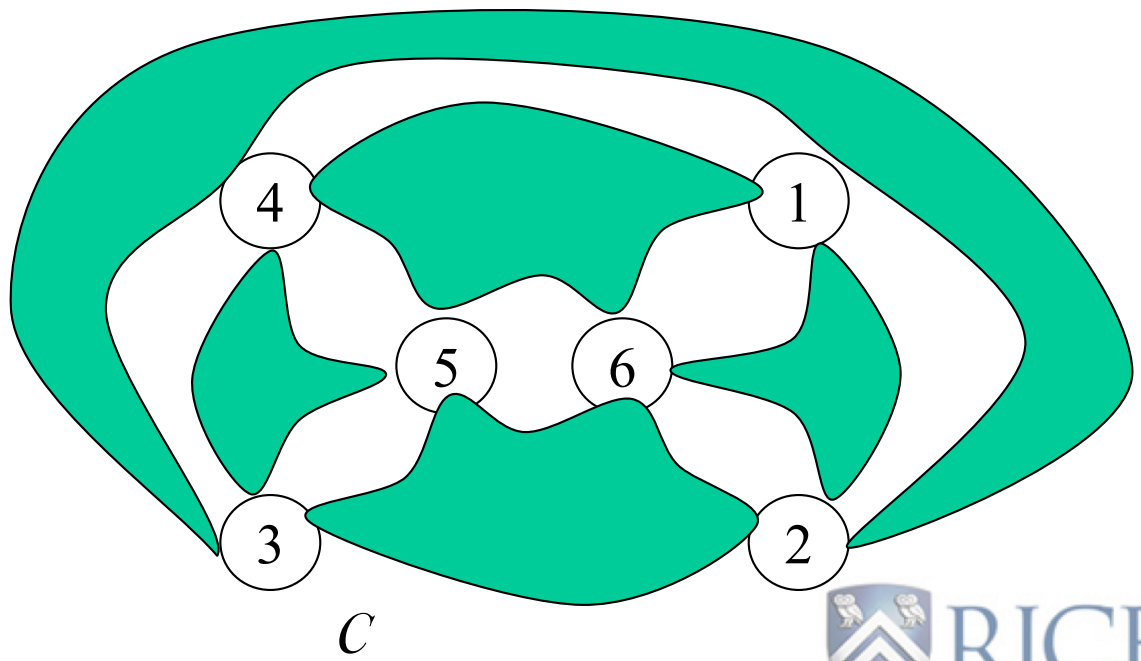




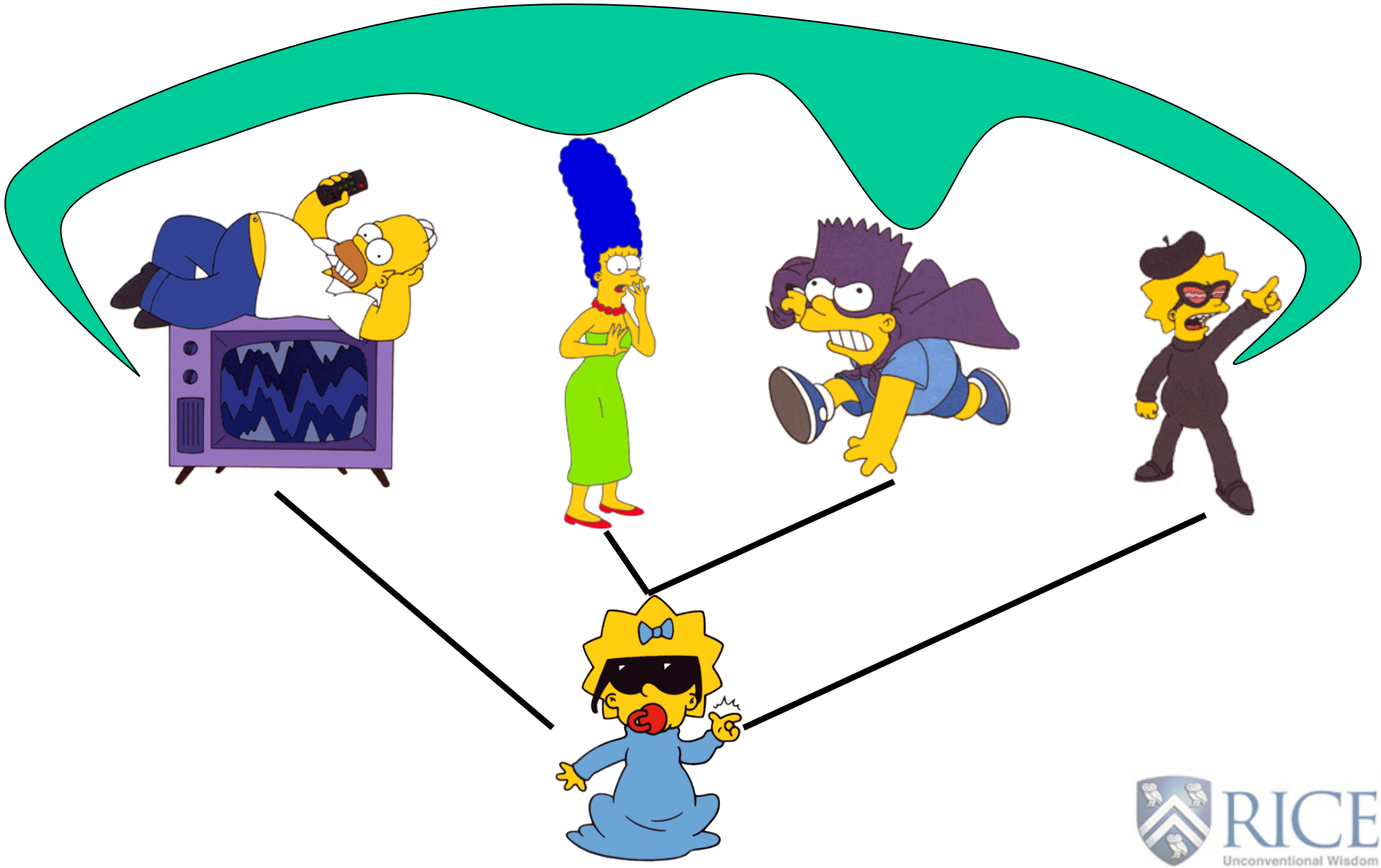
Clutters of Maximal 2-plexes

- Given a graph G, let C be the clutter whose vertices are $V(G)$ and whose edges are maximal 2-plexes of G.





Clutters of Maximal 2-plexes



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2-claw free & 2-plex Clutter Matrices

- Thm (McCloksy & Hicks 2009): A graph G is 2-claw free if and only if $\Delta(G) \leq 2$ or G is 2-plex.

- Thm (McClosky & Hicks, 2009): Let A be the 2-plex clutter matrix of G. The polytope $P=\{x \in R^n_+ : Ax \leq 2, x \leq 1\}$ is integral if and only if the components of G are 2-plexes, co-2-plexes, paths, or 0 mod 3 chordless cycles.

- Corollary (McClosky & Hicks, 2009): Given a 2-plex clutter matrix A, there is a polynomial-time algorithm to determine if $P=\{x \in R^n_+: Ax \leq 2, x \leq 1\}$ is integral.

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Analogs in Polyhedral Combinatorics

- K-plex Polytope
- Co-2-plex Polytope
- 2-Club Polytope (Pajouh, Balasundaram, Hicks, 2016

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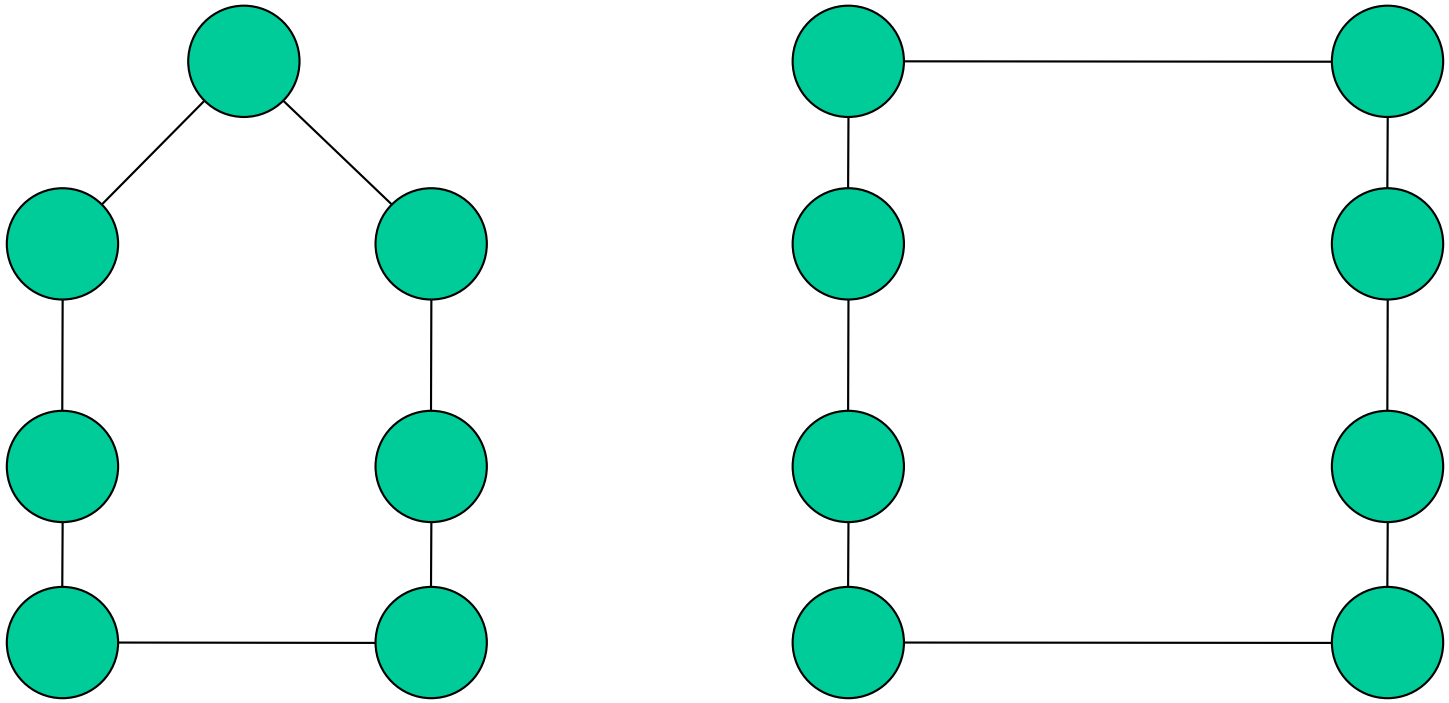
Analogs in Polyhedral Combinatorics for 2-club polytope

- Padberg (1973)
  - odd hole inequalities
  - Stable set inequalities

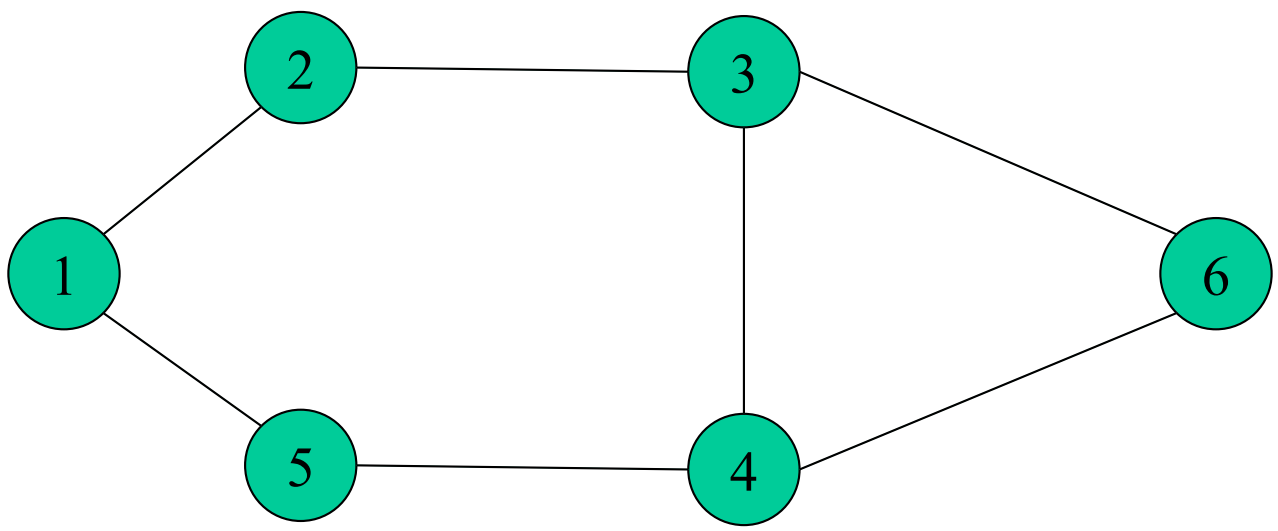
Odd Mod 3 Cycles



Thm: If C is an n-cycle such that $n > 6$ and $n \neq 0 \bmod 3$, then $\sum_{v \in V(C)} x_v \leq 3$ is a facet for 2-club polytope.

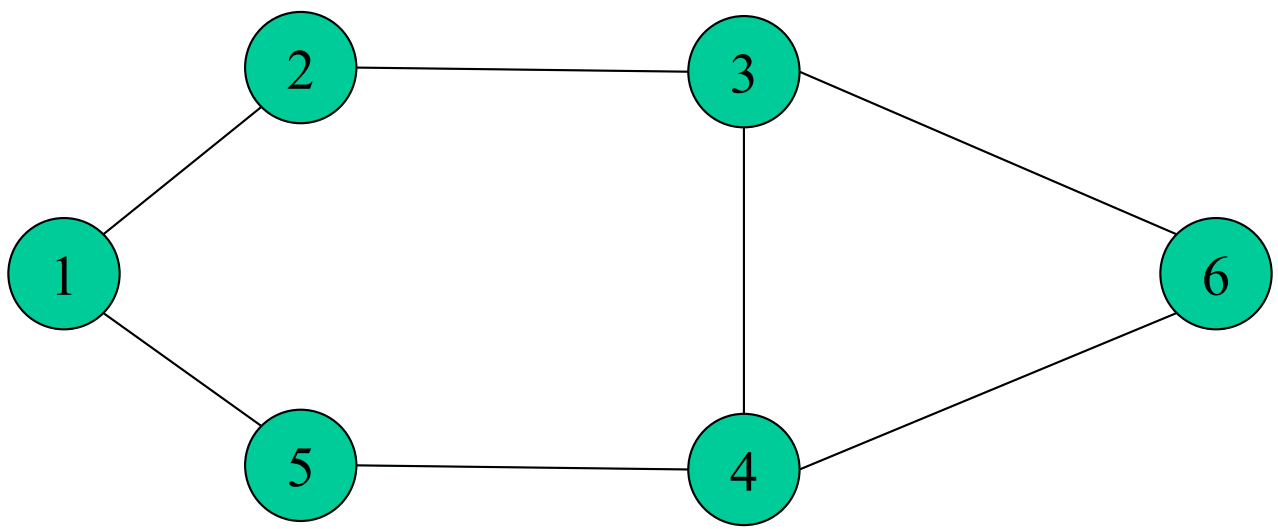
Stable Sets and Dominating Sets

- Given a graph $G=(V, E)$, a set $D \subseteq V$ is called a dominating set if for every node $v \in V \backslash D$ there exists $w \in D$ such that vw is an edge ($d(v,w) = 1$)
- A dominating set that is also a stable set is called an independent dominating set.



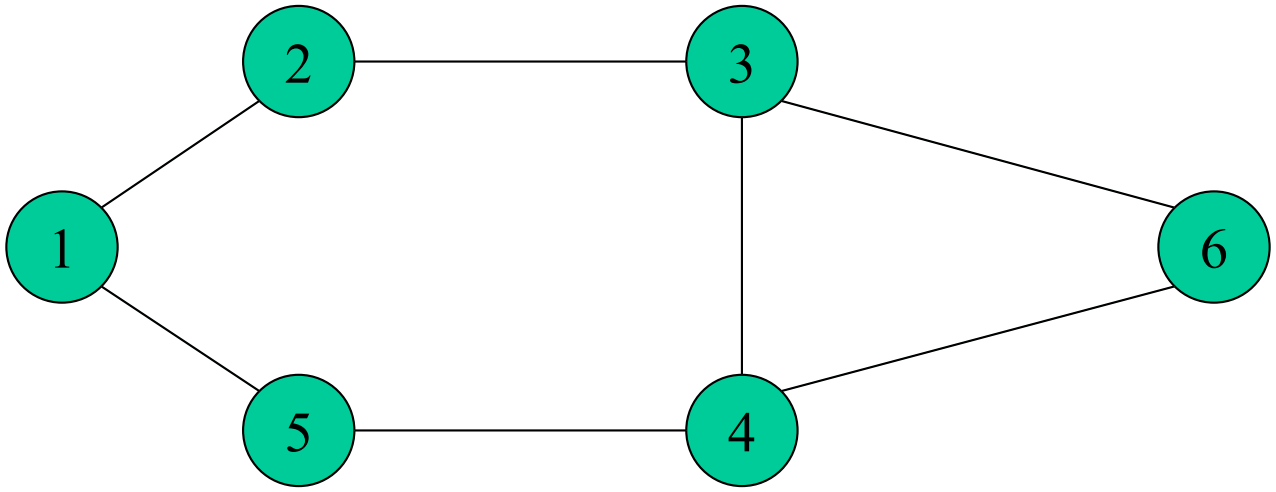
Stable Sets and Dominating Sets

- Theorem (Padberg 1973): If S is a maximal stable set in G, then $\sum_{v \in S} x_v \leq 1$ is a facet for P_1(G).
- Recall, Berge observed that a stable set is maximal if and only if it is dominating.



k-Dominating Set

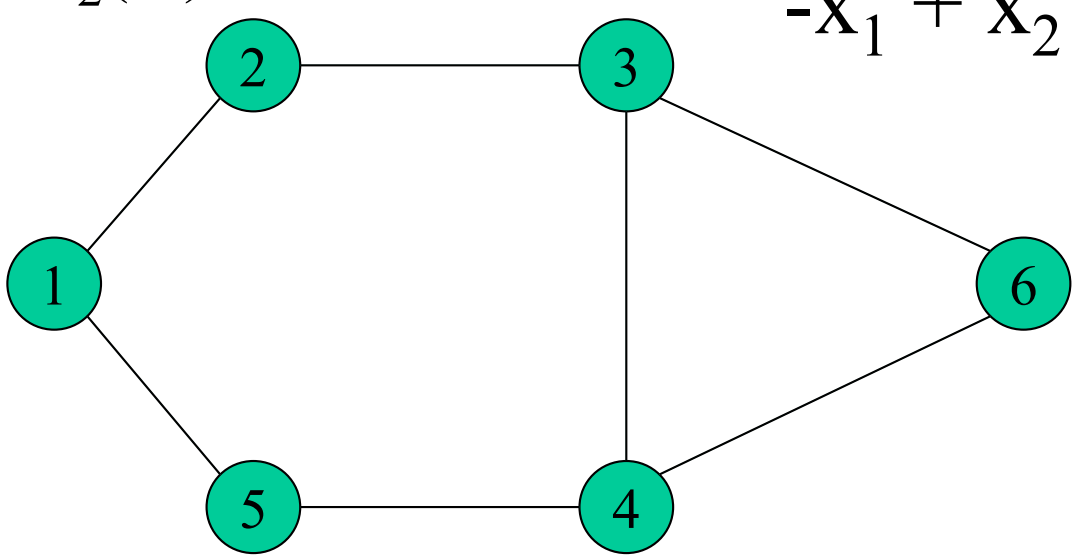
- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $D \subseteq V$ is called a *k*-dominating set if for every node $v \in V \backslash D$ there exists $w \in D$ such that $d(v,w) \leq k$
- A k-dominating set that is also independent is called an independent *k*-dominating set.



2-club Polytope: I2D inequalities

- Notation: $k^+ = \max\{k, 0\}$
- Theorem: Given a graph G=(V, E) and an independent 2-dominating (I2D) set $C \subseteq V$, then

$x(C) - \sum_{v \in V \setminus C} (|N(v) \cap C| - 1)^+ x_v \leq 1$ is facet inducing for $P_2(G)$.



$$-x_1 + x_2 - x_3 - x_4 + x_5 + x_6 \leq 1$$

$$-x_4 + x_5 + x_6 \leq 1$$

$$x_3 \leq 1$$

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I2D Facets

- Theorem: These I2D facets fully characterize 2-club polytope for acyclic graphs.
- All previous known non-trivial facets for the polytope are special cases of I2D inequalities
  - 2 independent sets (independent by a distance of 2) (Balasundaram et al. (2005))
  - Pair-blob (Carvalho et al. (2008))
  - Roof-blob (Carvalho et al. (2008))
- The separation problem for I2Ds is NP-hard.

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Wrap-Up

- Social Networks
- *k*-plexes & co-*k*-plexes & k-clubs
- Combinatorial algorithm analogs
- Polyhedral Combinatorics analogs

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Any Questions?

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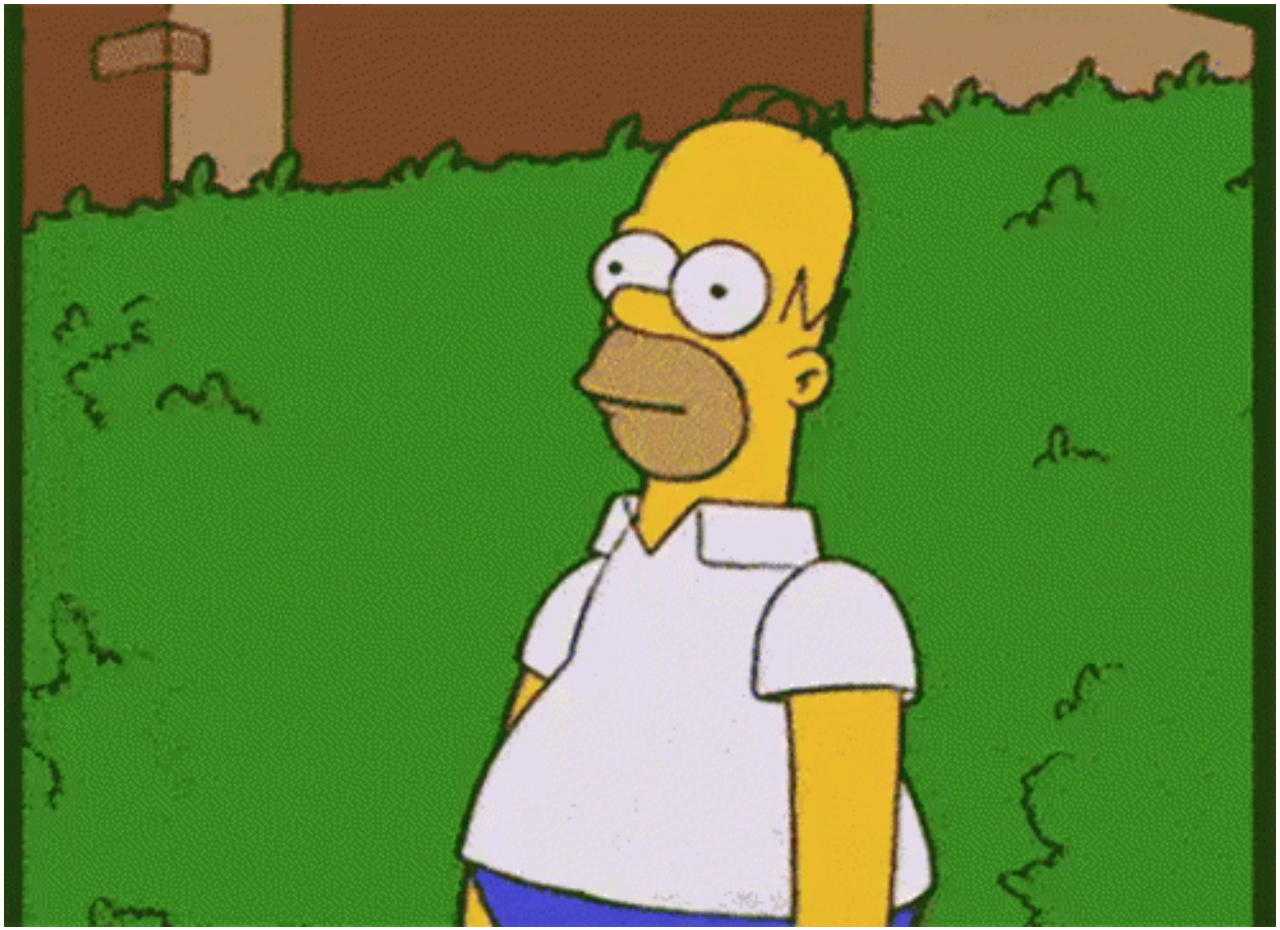
Time to Go



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